Wavelet bases on a triangle

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Abstract

In the present paper we find new constructions of orthonormal multiresolution analyses on the triangle Δ. In the first one, we describe a direct method to define an orthonormal multiresolution analysis $\mathbb{R}$ which is adapted for the study of the Sobolev spaces $H_0^s(\Delta)$ $(s \in \mathbb{N})$. In the second one, we add boundary conditions for constructing an orthonormal multiresolution analysis which is adapted for the study of the Sobolev spaces $H^s(\Delta)$ $(s \in \mathbb{N})$. The associated wavelets preserve the original regularity and are easy to implement.

Keywords : Multiresolution analysis, Wavelet, Sobolev space, Boundary condition, Norm equivalence.

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1 Introduction

Wavelets are functions generated from one basis function by dilations and translations. Wavelet concepts have unfolded their full computational efficiency mainly in harmonic analysis (for the study of Calderon-Zygmund operators) and in signal analysis. The wavelet expansions induce isomorphisms between function and sequence spaces. It means that certain Sobolev or Besov norms of functions are equivalent to weighted sequence norms for the coefficients in their wavelet expansions. The wavelets have cancellation properties that are usually expressed in terms of vanishing polynomial moments. The combination of the two previous properties of wavelets provides a rigorous analysis of adaptative schemes for elliptic problems. Moreover, nonlinear approximation is an important concept related to adaptative approximation.

The search for wavelet bases on a bounded domain has been an active field for many years, since the beginning of the nineties. Several strategies for dealing with complex domain geometries as manifolds have been explored in wavelet literature (see [5] and [8]). There are basically two approaches domain decomposition into cube patches and multilevel decomposition of finite element spaces. The first approach was introduced by Z. Ciesielski and T. Figiel in 1982 (see [3] and [4]) to construct spline bases of generalized Sobolev spaces $W_p^k(M)$ ($k \in \mathbb{Z}$ and $1 < p < \infty$) where M is a compact Riemannian manifold. This method is based on a characterization of function spaces on manifolds as products of corresponding local function spaces subject to certain complementary boundary conditions. It was adapted to the wavelet setting to construct generalized multiresolution analyses on bounded domains. The decomposition method turns out to have principal limitations and it does not induce Sobolev Spaces H^s when $|s| \geq 3/2$. The basic difficulty is that function spaces as Sobolev or Besov spaces on compact Riemannian manifolds (with or without boundary) are usually defined in terms of open covering and associated charts, not in terms of partitions of the manifold. The second approach can be more tempting if one wants to combine properties of wavelet bases (cancellation, dilation, translation) with the structural simplicity of finite element spaces.

In 1992, A. Jouini and P.G. Lemarié constructed in [13] biorthogonal wavelet bases on two-dimensional manifold Ω; these bases were adapted for the study of Sobolev spaces $H^1(\Omega)$ and $H^1_0(\Omega)$. In 1997, the decomposition method was used by C. Cohen, W. Dahmen and R. Schneider in ([5], [7] and [8]) to construct biorthogonal wavelet bases $(\psi_\lambda, \widetilde{\psi}_\lambda)_{\lambda\in\nabla}$ of $L^2(\Omega)$ where Ω is a bounded domain of $\mathbb{R}^d$ ($d \in \mathbb{N}^*$); these bases were shown to be bases of Sobolev spaces $H^s(\Omega)$ for $-\frac{3}{2} < s < \frac{3}{2}$. There are related constructions as well by C. Canuto and coworkers in [2] and by R. Masson in [17]. All these constructions are based on the decomposition method by gluing scaling functions and wavelets across the interfaces of adjacent subdomains. These bases are continuous but not differentiable; there is a slight difficulty in their presentation, due to notational burden. Moreover, it is often unclear how to get other regularity Sobolev estimates than for $-\frac{3}{2} < s < \frac{3}{2}$. In 2003, A. Jouini and P.G. Lemarié presented in [14] two elementary constructions of orthogonal and biorthogonal wavelet bases on the L-shaped domain L. In the first one, they used a direct method and in the second one, they used a decomposition method. These bases have simple expressions and the specific geometry of the domain allows to get higher regularity namely the study of the Sobolev spaces $H^k(L)$ ($k \in \mathbb{Z}$). Recently, in 2008, A. Jouini and M. Kratou used decomposition method to construct in [11] biorthogonal wavelet bases on a compact Riemannian manifold with dimension n (or an open bounded set of $\mathbb{R}^n$), $n \in \mathbb{N}^*$. The central problem was to construct extension operators which are straightforward, relatively simple and are adapted to the scale. This construction of biorthogonal analyses differed from the previous one in the sense that these analyses are generated by a finite number of simple basic functions and had better stability constants. They were also adapted for the study of the Sobolev spaces H^1 and H^1_0.

The construction of wavelet bases on the triangle has not been extensively discussed in the literature. Nevertheless, the triangle is used in wavelet theory to define triangulation of resolution 2^{-j} for general domains in the two-dimensional case. However, we do not have on the triangle regular bases with compact support and which are simple to implement. The central problem is to construct wavelet spaces. Moreover, we do not have general criteria available in literature that tell under which circumstances one has norm equivalences and how to conclude stability on a special domain as the triangle.

In this paper, we use a direct method to construct two orthogonal multiresolution analyses on a triangle Δ which are adapted to higher regularity analysis. The scaling spaces are constructed in an elementary way. Our construction does not involve domain decomposition at all but uses boundary adaptation for fairly general lipschitz domains. The main contribution offered in this paper which differs from the other constructions is the realization of global higher regularity by more elementary techniques than perhaps those involved in ([5], [8] and [11]).
Section 2 is devoted to the description of multiresolution analyses on the interval. We develop two ways for constructing wavelet bases which will be useful for the remainder of the work.

In section 3, we shall use a direct method based on the results described in ([12] and [18]) to define two orthonormal multiresolution analyses on a triangle Δ.
In section 4, we prove a regularity lemma which gives uniform estimates for extension operators on the scaling spaces. This Lemma is very important to establish norm equivalences in Sobolev spaces on the triangle.
In the last section, we study and construct in the first part the associated wavelet bases on the triangle Δ. This construction is complicated and technical due to the geometry of the domain. In the second part and as applications, we characterize regular spaces namely Sobolev spaces $H_0^s(\Delta)$ and $H^s(\Delta)$ ($s \in \mathbb{N}$) in terms of discrete norm equivalences. At the end of this work, we consider the cases $N = 1$ (Haar basis) and $N = 2$. These examples permit to illustrate the constructions of wavelet bases of this paper and to explain clearly the relation between the support of the wavelet and the geometry of the domain.

We recall that all bases constructed in this work have compact support and the same regularity as for Daubechies bases [9].

NOTATIONS. We denote by
-MRA : Multiresolution analysis
-OMRA : Orthogonal multiresolution analysis.

2 Multiresolution analysis on the interval

Let us recall that I. Daubechies constructed in [9] an orthogonal multiresolution analysis $V_j(\mathbb{R})$ of $L^2(\mathbb{R})$ satisfying the following properties:

- $V_0(\mathbb{R})$ has an orthonormal basis $\varphi(x-k)$, $k \in \mathbb{Z}$, where φ the scaling function with compact support.
- $\varphi(\frac{x}{2}) = \sum_{k=0}^{2N-1} a_k \varphi(x-k)$, the sequence of real numbers (a_k) satisfies $a_0 \neq 0$ and $a_{2N-1} \neq 0$. Moreover, we have $\hat{\varphi}(2\xi) = M_0(e^{-i\xi})\hat{\varphi}(\xi)$ where $M_0(e^{-i\xi}) = \sum_{k=0}^{2N-1} a_k e^{-ki\xi}$ and " $\hat{f}$ " is the classical Fourier transform of f on $\mathbb{R}$.
- $V_j(\mathbb{R})$ has an orthonormal basis $\varphi_{j,k}(x) = 2^{\frac{j}{2}}\varphi(2^j x - k)$, j, $k \in \mathbb{Z}$.
- $\mathrm{Supp}\varphi = [0, 2N-1]$.
- φ has Sobolev regularity H^{s_N} with $s_N = (1 - \frac{\ln 3}{\ln 4})N + o(N)$.
- The associated wavelet ψ is defined by

$$\hat{\psi}(2\xi) = e^{-i(2N-1)\xi} M_0(e^{-i\xi})\hat{\varphi}(\xi).$$

- $W_0(\mathbb{R})$ (the orthogonal complement of $V_0(\mathbb{R})$ in $V_1(\mathbb{R})$) has an orthonormal basis $\psi(x-k)$, $k \in \mathbb{Z}$ and $W_j(\mathbb{R})$ has an orthonormal basis $\psi_{j,k}(x) = 2^{\frac{j}{2}}\psi(2^j x - k)$, j, $k \in \mathbb{Z}$.
- The moments of the related wavelet ψ satisfy $\int x^k \psi(x)dx = 0$ for $0 \leq k \leq N-1$.

It is well known that we can have orthonormal wavelet bases which allow synthesis of more general functional spaces than $L^2(\mathbb{R})$. Such a fact was not true by restriction to a subset of $\mathbb{R}$ as simple as the set [0,1]. The problem is that, in bounded domains, classical invariance by dilation and translation are preserved for dilation, on the other hand they lose in part their meaning for translation. Moreover, the multiresolution analysis of I. Daubechies is orthogonal in $L^2(\mathbb{R})$, but if we take its restriction to $[0,1]$, we do not get an orthogonal multiresolution analysis in $L^2([0,1])$. If we consider the functions $\varphi_{j,k}(x)_{/[0,1]}$, we have a linearly independent system but not orthogonal. However, if we consider the functions $\psi_{j,k}(x)_{/[0,1]}$, we get a linearly dependent system (see [18]). Then, the construction of orthogonal multiresolution analyses in $[0,1]$ (or biorthogonal) is technical specially near the boundaries 0 and 1.

In the following, we consider the OMRA $V_j(\mathbb{R})$ of I. Daubechies and we denote

$$V_j([0,1]) = Span\{\varphi_{j,k/[0,1]}, \varphi_{j,k} \in V_j(\mathbb{R})\} \tag{2.1}$$

and

$$v_j([0,1]) = Span\{\varphi_{j,k}, supp\varphi_{j,k} \subset [0,1]\}. \tag{2.2}$$

Remark 2.1 *If I is a bounded interval of $\mathbb{R}$, the space $V_j(I)$ is defined as the space of restrictions to I of elements of $V_j(\mathbb{R})$. More precisely, we may keep only the indexes k such that $(2^{-j}k, 2^{-j}(k+2N-1)) \cap I \neq \emptyset$.*

Recall the following important result.

lemma 2.1 *Let f be a scaling function with minimal support, then the restrictions $(f(x-k)_{/[0,1]} \neq 0)$ are linearly independent.*

This Lemma is proved in [15]. This result states in particular that the restrictions $\varphi(x-k)_{/[0,1]}$ are linearly independent for $2-2N \leq k \leq 0$.

Remark 2.2 *Let $I = [\alpha, \beta]$. For $j \in \mathbb{Z}$, let α_j the smallest integer which is greater than $2^j\alpha - 2N + 1$ and let β_j the greatest integer which is smaller than $2^j\beta$. The functions $(\varphi_{j,k})_{/I}$, $\alpha_j \leq k \leq \beta_j$ are linearly independent, and thus they are a basis for $V_j(I)$.*

Remark 2.3 *Under the assumptions of Remark 2.2, there exists a constant $c(j, I)$ such that for all sequences $(\lambda_k)_{\alpha_j \leq k \leq \beta_j}$ we have the inequality*

$$c(j, I) \sum_{\alpha_j \leq k \leq \beta_j} |\lambda_k|^2 \leq \int_\alpha^\beta |\sum_{k \in \mathbb{Z}^2} \lambda_k \varphi_{j,k}|^2 dx \leq \sum_{\alpha_j \leq k \leq \beta_j} |\lambda_k|^2. \tag{2.3}$$

If α or β is not a dyadic number, we may have lim $inf_{j \to +\infty} c(j, I) = 0$: we have $c(j, I) \leq \min(\int_\alpha^{2^{-j}\alpha_j} |\varphi|^2 dx, \int_{2^{-j}\beta_j}^\beta |\varphi|^2 dx)$. On the other hand, when α and β are dyadic numbers, $c(j, I)$ does not depend on j when j is big enough.

The existence of an orthonormal basis of $L^2([0, 1])$ allowing the characterization of regular function on the interval [0,1] and having simple algorithms was treated by Y. Meyer in [18]. Starting from the orthogonal multiresolution analysis of I. Daubechies, Y. Meyer introduced the orthogonal projection operators P_j from $L^2([0, 1])$ onto $V_j([0, 1])$ and $Q_j = P_{j+1} - P_j$ and he showed that the ranges of $V_j = ImP_j$ and $W_j = ImQ_j$ have elementary Hilbertian bases. This is based on the remark that, while $V_{j+1}([0, 1])$ is generated by the restrictions of the scaling functions $\varphi_{j,k}$ and the wavelets $\psi_{j,k}$, $-2N + 2 \leq k \leq 2^j - 1$, the restrictions of the extreme wavelets $\psi_{j,k}$, $-2N + 2 \leq k \leq -N$ and $2^j - 2N + 1 \leq k \leq 2^j - 1$ belong to $V_j([0, 1])$, so their elimination gives a generating system of $(2^{j+1} + 2N - 2)$ vectors of $V_{j+1}([0, 1])$, hence we have the following Meyer's Lemma.

lemma 2.2 *Let j_0 be the smallest integer satisfying $2^{j_0} \geq 4N - 4$. Then, for $j \geq j_0$, the functions $\varphi_{j,k/[0,1]}$, $2 - 2N \leq k \leq 2^j - 1$, (which form a Riesz basis for $V_j([0, 1])$) and the functions $\psi_{j,k/[0,1]}$, $-N + 1 \leq k \leq 2^j - N$, constitute a Riesz basis for $V_{j+1}([0, 1])$.*

Clearly, the inclusion $V_j(\mathbb{R}) \subset V_{j+1}(\mathbb{R})$ gives $V_j([0, 1]) \subset V_{j+1}([0, 1])$. Thus, we may define $W_j([0, 1]) = (V_j([0, 1]))^\perp \cap V_{j+1}([0, 1])$. A Riesz basis for $V_j([0, 1])$ is given by the restriction of functions $\varphi_{j,k}$, $-2N + 2 \leq k \leq 2^j - 1$; this basis can be split into three subfamilies: the functions located at the border 0 ($(\varphi_{j,k})_{/[0,1]}$, $-2N + 2 \leq k \leq -1$), the functions which have their support included in $[0, 1]$ ($\varphi_{j,k}$, $0 \leq k \leq 2^j - 2N + 1$) and the functions located at the border 1 ($(\varphi_{j,k})_{/[0,1]}$, $2^j - 2N + 2 \leq k \leq 2^j - 1$). Orthonormalization of the first family and the third one then gives an orthonormal basis $\phi_{j,k}$, $-2N + 2 \leq k \leq 2^j - 1$ with $\phi_{j,k} = 2^{\frac{j}{2}}\varphi(2^j x - k)$ for $0 \leq k \leq 2^j - 2N + 1$, $\phi_{j,k} = 2^{\frac{j}{2}}\varphi_l^{[q]}(2^j x)$ for $1 \leq q \leq 2N-2$ and $k = -q$ and $\phi_{j,k} = 2^{\frac{j}{2}}\varphi_r^{[q]}(2^j(x-1))$ for $1 \leq q \leq 2N - 2$ and $k = 2^j - 2N + q + 1$, where the functions $\varphi_l^{[q]}$ have a compact support in $[0, +\infty)$ and the functions $\varphi_r^{[q]}$ have a compact support in $(-\infty, 0]$.

The Riesz basis for $V_j([0, 1])$ may then be completed into a Riesz basis for $V_{j+1}([0, 1])$ by adding the restrictions of the wavelets $\psi_{j,k}$, $-N+1 \leq k \leq 2^j - N$. This new set may be split into three subfamilies, the functions located at the border 0 ($(\psi_{j,k})_{/[0,1]}$, $-N + 1 \leq k \leq -1$), the functions which have their support included in $[0, 1]$ ($\psi_{j,k}$, $0 \leq k \leq 2^j - 2N + 1$) and the functions located at the border 1 ($(\psi_{j,k})_{/[0,1]}$, $2^j - 2N + 2 \leq k \leq 2^j - N$). The second family already belongs to $W_j([0, 1])$, while we need corrections on the first and third ones. We then get an orthonormal basis $\varpi_{j,k}$, $-N+1 \leq k \leq 2^j - N$ for $W_j([0, 1])$, with $\varpi_{j,k} = 2^{\frac{j}{2}}\psi(2^j x - k)$ for $0 \leq k \leq 2^j - 2N + 1$, $\varpi_{j,k} = 2^{\frac{j}{2}}\psi_l^{[q]}(2^j x)$ for $1 \leq q \leq N - 1$ and $k = -q$ and $\varpi_{j,k} = 2^{\frac{j}{2}}\psi_r^{[q]}(2^j(x - 1))$ for $1 \leq q \leq N - 1$ and $k = 2^j - 2N + q + 1$, where the functions $\psi_l^{[q]}$ have a compact support in $[0, +\infty)$ and the functions $\psi_r^{[q]}$ have a compact support in $(-\infty, 0]$.

Meyer's basis on the interval is theoretically important and constitutes a fundamental starting point for other constructions. Cohen, Daubechies, Vial and Jawerth constructed at 1992 in [6] a multiresolution analysis on the interval in such manner that they got a polynomial extension outside the interval. Jouini and Lemarié defined at 1993 in [12] a multiresolution analysis on the interval as the following.

Definition 2.1 *A sequence $\{V_j\}_{j\geq j_0}$ of closed subspaces of $L^2([0, 1])$ is called a MRA on $L^2([0, 1])$ associated with $V_j(\mathbb{R})$ if we have*

i) $\forall j \geq j_0$, $v_j([0, 1]) \subset V_j \subset V_j([0, 1])$

ii) $\forall j \geq j_0$, $V_j \subset V_{j+1}$.

Starting again from the orthogonal multiresolution analysis of I. Daubechies, they proposed a new wavelet space $W_j([0, 1])$ by reviewing the collection of the functions on the border. More precisely, we have the second important result from [12].

Proposition 2.1 *Let j_0 be the smallest integer satisfying $2^{j_0} \geq 4N-4$. For $j \geq j_0$, we denote*

$$\begin{aligned} X_j \quad &= Span\{\psi_{j,k}, 0 \leq k \leq 2^j - 2N+1; \varphi_{j+1,2k+1}, \\ &0 \leq k \leq N-2; \varphi_{j+1,2k}, 2^j - 2N+2 \leq k \leq 2^j - N\}. \end{aligned} \tag{2.4}$$

Then,

i) dim $X_j = 2^j$

ii) there exists an integer J such that $\forall j \geq J$, $V_{j+1} = V_j \oplus X_j$.

It is clear that we can realize orthogonality by using Gram-Schmidt.

3 The spaces $v_j(\Delta)$ and $V_j(\Delta)$

As usually in wavelet theory, we define $V_j(\mathbb{R}^2)$ the multiresolution analysis associated to the separable scaling function $\varphi \otimes \varphi$: $V_j(\mathbb{R}^2)$ is the tensor product $V_j(\mathbb{R}^2) = V_j(\mathbb{R})\widehat{\otimes} V_j(\mathbb{R})$. For a generic domain Ω, we cannot expect a simple description of the space $V_j(\Omega)$: even in the univariate case and in the case of an elementary interval, we have difficulties in estimating the bases. On the other hand, for very simple cases, we get an easy description. As an example, if we consider the unit cube $[0,1]^n$, then, by using Lemma 2.1 and tensor product, we get a direct basis of $V_j([0,1]^n)$ (see [14]).

The next domain we shall consider is the triangle $\Delta = \{(x,y) \in [-1,1] \times [0,1],\ y \leq 1-|x|\}$. In the following, we study a multiresolution analysis on Δ without boundary conditions.

Definition 3.1 *The space $v_j(\Delta)$ is defined as the space of elements of $V_j(\mathbb{R}^2)$ with support in Δ.*

We can describe directly an orthonormal basis of $v_j(\Delta)$.

Proposition 3.1 *For $2^j \geq 4N-4$, $v_j(\Delta)$ has the following basis: the family $\varphi_{j,k_1} \otimes \varphi_{j,k_2}$ with $-2^j+2N-1 \leq k_1 \leq -N$, $0 \leq k_2 \leq k_1+2^j-2N+1$ and the family $\varphi_{j,k_1} \otimes \varphi_{j,k_2}$ with $-N+1 \leq k_1 \leq 2^j-4N+2$, $0 \leq k_2 \leq -k_1+2^j-4N+2$.*

If we look at our domain and the support of the scaling function, we can split these families into the following sets:

i) left functions: $\varphi_{j,k_1} \otimes \varphi_{j,k_2}$ with $-2^j + 2N - 1 \leq k_1 \leq -N$ and $0 \leq k_2 \leq k_1 + 2^j - 2N + 1$

ii) right functions: $\varphi_{j,k_1} \otimes \varphi_{j,k_2}$ with $-N + 1 \leq k_1 \leq 2^j - 4N + 2$ and $0 \leq k_2 \leq -k_1 + 2^j - 4N + 2$.

The orthogonal projection operator on $v_j(\Delta)$ is written P_j^0 and the projection operator Q_j^0 on $w_j(\Delta) = (v_j(\Delta))^\perp \cap v_{j+1}(\Delta)$ is given by $Q_j^0 = P_{j+1}^0 - P_j^0$. Thus, P_j^0 is given by

$$\begin{aligned} P_j^0 f \; &= \sum_{k_1=-2^j+2N-1}^{-N} \sum_{k_2=0}^{k_1+2^j-2N+1} < f/\varphi_{j,k_1} \otimes \varphi_{j,k_2} > \varphi_{j,k_1} \otimes \varphi_{j,k_2} \\ &+ \sum_{k_1=-N+1}^{2^j-4N+2} \sum_{k_2=0}^{-k_1+2^j-4N+2} < f/\varphi_{j,k_1} \otimes \varphi_{j,k_2} > \varphi_{j,k_1} \otimes \varphi_{j,k_2}. \end{aligned} \tag{3.1}$$

Our main goal below will be a study of regular functions on the triangle with trace on the border. We introduce now the second orthogonal multiresolution analysis on the triangle Δ with boundary conditions. This point poses no problem for generator basic functions of scaling spaces.

Definition 3.2 *The space $V_j(\Delta)$ is defined as the space of restrictions to Δ of elements of $V_j(\mathbb{R}^2)$.*

We have an obvious generating family of $V_j(\Delta)$.

Proposition 3.2 *For $2^j \geq 4N - 4$, $V_j(\Delta)$ has the following basis: the family $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $k_1 = -2^j - 2N + 2 + p$, $0 \leq p \leq 2^j - 2$ and $-2N + 2 \leq k_2 \leq p$; the family $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $k_1 = -2N + 1 + p$, $0 \leq p \leq 2N - 1$ and $-2N + 2 \leq k_2 \leq 2^j - 1$ and the family $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $k_1 = 1 + p$, $0 \leq p \leq 2^j - 2$ and $-2N + 2 \leq k_2 \leq 2^j - 2 - p$.*

It is clear that Lemma 2.1 and Remark 2.2 prove that the system described in Proposition 3.2 is linearly independent. Then, the spaces $V_j(\Delta)$ define an orthogonal multiresolution analysis on Δ. The orthogonal projection operator on $V_j(\Delta)$ is written P_j and the projection operator Q_j on $W_j(\Delta) = (V_j(\Delta))^\perp \cap V_{j+1}(\Delta)$ is given by $Q_j = P_{j+1} - P_j$.

We shall now give a precise description of P_j by giving an orthonormal basis for $V_j(\Delta)$.

If we look at our domain, we can split these families into the following sets:

i) left functions: $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $k_1 = -2^j - 2N + 2 + p$, $0 \leq p \leq 2^j - 2$ and $-2N + 2 \leq k_2 \leq p$

ii) center functions: $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $k_1 = -2N + 1 + p$, $0 \leq p \leq 2N - 1$ and $-2N + 2 \leq k_2 \leq 2^j - 1$

iii) right functions: $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $k_1 = 1 + p$, $0 \leq p \leq 2^j - 2$ and $-2N + 2 \leq k_2 \leq 2^j - 2 - p$.

See that the number of left functions is equal to the number of right functions and this is due to symmetry of our domain. If we look now at the supports of these functions, we can split these families into the following sets:

i) interior functions: $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $-2^j + 2N - 1 \leq k_1 \leq -N$ and $0 \leq k_2 \leq k_1 + 2^j - 2N + 1$; $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $-N + 1 \leq k_1 \leq 2^j - 4N + 2$ and $0 \leq k_2 \leq -k_1 + 2^j - 4N + 2$

ii) edge functions: $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $-2^j - 2N + 3 \leq k_1 \leq -2^j + 2N - 2$ and $1 \leq k_2 \leq k_1 + 2^j + 2N - 2$; $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $-2^j + 2N - 1 \leq k_1 \leq 2^j - 4N + 2$ and $2 - 2N \leq k_2 \leq -1$; $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $-2^j + 2N - 1 \leq k_1 \leq -2N$ and $k_1 + 2^j - 2N + 2 \leq k_2 \leq k_1 + 2^j + 2N - 2$; $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $-2N + 1 \leq k_1 \leq -N - 1$ and $k_1 + 2^j - 2N + 2 \leq k_2 \leq -k_1 + 2^j - 4N + 1$; $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $-N + 2 \leq k_1 \leq 0$ and $-k_1 + 2^j - 4N + 3 \leq k_2 \leq k_1 + 2^j - 2N$; $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $1 \leq k_1 \leq 2^j - 4N + 2$ and $-k_1 + 2^j - 4N + 3 \leq k_2 \leq -k_1 + 2^j - 1$; $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $2^j - 4N + 3 \leq k_1 \leq 2^j - 2$ and $1 \leq k_2 \leq -k_1 + 2^j - 1$

iii) exterior corner functions: $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $-2^j - 2N + 2 \leq k_1 \leq -2^j + 2N - 2$ and $2 - 2N \leq k_2 \leq 0$; $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $2^j - 4N + 3 \leq k_1 \leq 2^j - 1$ and $2 - 2N \leq k_2 \leq 0$

iv) interior corner functions: $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $-2N + 1 \leq k_1 \leq -N$ and $-k_1 + 2^j - 4N + 2 \leq k_2 \leq 2^j - 1$; $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $-N + 1 \leq k_1 \leq 0$ and $k_1 + 2^j - 2N + 1 \leq k_2 \leq 2^j - 1$.

Orthonormalization of the border functions (edge and corner functions) gives an orthonormal basis $\phi_{j,k_1,k_2/\Delta}$ for $V_j(\Delta)$ where $(k_1, k_2) \in M_j$ (to simplify notations) and $\mathrm{card} M_j = \dim V_j(\Delta) = 2^{2j} + (6N - 5)2^j + (2N - 2)^2$. All these functions are regular (same regularity as Daubechies scale function). Thus, we have the orthogonal projection operator P_j onto $V_j(\Delta)$ as

$$P_j f = \sum_{(k_1,k_2)\in M_j} < f/\phi_{j,k_1,k_2/\Delta} > \phi_{j,k_1,k_2/\Delta}. \tag{3.2}$$

4 A regularity lemma

We prove in this section some technical lemmas which will be useful in regularity analysis for functions defined on Δ.

lemma 4.1 *Let w_1, w_2 be two square integrable compactly supported functions on $\mathbb{R}^2$. Then the operator $f \to \sum_{k\in\mathbb{Z}^2} < f/w_1(.-k) > w_2(.-k)$ is bounded on $L^2(\mathbb{R})$.*

Proof. Let $L \in \mathbb{N}$ be such that the supports of w_1 and w_2 are contained in $(-L,L)^2$. Then we have

$$\begin{aligned}\int \left|\sum_{k\in\mathbb{Z}^2} \lambda_k w_2(x-k)\right|^2 dx &\le 4L^2 \int \sum_{k\in\mathbb{Z}^2} |\lambda_k w_2(x-k)|^2 dx \\ &= 4L^2 \|w_2\|_2^2 \sum_{k\in\mathbb{Z}^2} |\lambda_k|^2\end{aligned}$$

and

$$\begin{aligned}\sum_{k\in\mathbb{Z}^2} \left|\int f(y) w_1(y-k) dy\right|^2 &\le \|w_1\|_2^2 \sum_{k\in\mathbb{Z}^2} \int |f(y)|^2 1_{(-L,L)^2}(y-k) dy \\ &\le 4L^2 \|w_1\|_2^2 \|f\|_2^2.\end{aligned}$$

Thus, the lemma is obvious. ■

Definition 4.1 *Let $\Delta = \{(x,y) \in [-1,1]\times[0,1],\ y \le 1-|x|\}$. Let us consider, for $2^j \ge 4N-4$, the basis for $V_j(\Delta)$ given by the family $(\phi_{j,k_1,k_2/\Delta})_{(k_1,k_2)\in M_j}$ described in (3.2). Then we define the extension operator E_j from $V_j(\Delta)$ to $V_j(\mathbb{R}^2)$ by the formula*

$$E_j f = \sum_{(k_1,k_2)\in M_j} < f/\phi_{j,k_1,k_2/\Delta} >_\Delta \phi_{j,k_1,k_2}, \tag{4.1}$$

where $< f/g >_\Delta = \int_\Delta f\overline{g}dx$.

We establish now the first main result of this section. In fact, the following lemma is very useful for analyzing regular functions on the triangle.

lemma 4.2 *Let $\Delta = \{(x,y) \in [-1,1]\times[0,1],\ y \le 1-|x|\}$. There exists a positive constant α such that for all j such that $2^j \ge 4N-4$ and all $f \in V_j(\Delta)$:*

$$\|E_j f\|^2_{L^2(\mathbb{R}^2)} \le \alpha \|f\|^2_{L^2(\Delta)}. \tag{4.2}$$

Proof. We divide $[0,1]\times[0,1]$ into four triangles defined by: for $0 \leq \eta \leq 3$, $T^\eta = \{(x,y) \in [0,1]\times[0,1]/(-1)^\eta(x-y) \geq 0$ and $(-1)^{E(\frac{\eta}{2})}(x+y-1) \geq 0\}$.

We triangulate $\mathbb{R}^2$ such that

$$\mathbb{R}^2 = \bigcup_{0\leq\eta\leq 3} \bigcup_{(k_1,k_2)\in\mathbb{Z}^2} T^\eta_{j,k_1,k_2}$$

where

$$T^\eta_{j,k_1,k_2} = \{(x,y)/(2^j x - k_1, 2^j y - k_2) \in T^\eta\}.$$

This triangulation is adapted to our triangle $\Delta = \{(x,y) \in [-1,1] \times [0,1], y \leq 1-|x|\}$ because we have

$$\Delta = \bigcup_{T^\eta_{j,k_1,k_2}\subset\Delta} T^\eta_{j,k_1,k_2}.$$

We put $\phi_{j,k_1,k_2}(x,y) = 2^j\varphi(2^j x - k_1)\varphi(2^j y - k_2)$ and $\phi_{k_1,k_2} = \phi_{0,k_1,k_2}$. Let us write

$$\int\int_\Delta |\sum_{(k_1,k_2)\in\mathbb{Z}^2} \alpha_{k_1,k_2}\phi_{j,k_1,k_2}|^2 dxdy = \sum_{T^\eta_{j,l_1,l_2}\subset\Delta} \int\int_{T^\eta_{j,l_1,l_2}} |\sum_{(k_1,k_2)\in\mathbb{Z}^2} \alpha_{k_1,k_2}\phi_{j,k_1,k_2}|^2 dxdy;$$

then,

$$\int\int_{T^\eta_{j,l_1,l_2}} |\sum_{(k_1,k_2)\in\mathbb{Z}^2} \alpha_{k_1,k_2}\phi_{k_1,k_2}|^2 dxdy = \int\int_{T^\eta} |\sum_{(k_1,k_2)\in\mathbb{Z}^2} \alpha_{k_1,k_2}\phi_{k_1-l_1,k_2-l_2}|^2 dxdy.$$

Let C^η be the set of indexes (k_1,k_2) such that the support of ϕ_{k_1,k_2} has an intersection of non vanishing measure with T^η, C^η_{j,k_1,k_2} the set of indexes (l_1,l_2) such that the support of ϕ_{j,l_1,l_2} has an intersection of non vanishing measure with T^η_{j,k_1,k_2} and C_j the set of indexes (k_1,k_2) such that the support of ϕ_{j,k_1,k_2} has an intersection of non vanishing measure with Δ. We have $C_j = \bigcup_{T^\eta_{j,k_1,k_2}\subset\Delta} C^\eta_{j,k_1,k_2}$. The family $(\phi_{k_1,k_2|T^\eta})_{(k_1,k_2)\in C^\eta}$ is linearly independent. Then, there exists a positive constant γ such that we have

$$\int\int_{T^\eta} |\sum_{(k_1,k_2)\in\mathbb{Z}^2} \beta_{k_1,k_2}\phi_{k_1,k_2}|^2 dxdy \geq \gamma \sum_{(k_1,k_2)\in C^\eta} |\beta_{k_1,k_2}|^2;$$

hence,

$$\int\int_{T^\eta_{j,l_1,l_2}} |\sum_{(k_1,k_2)\in\mathbb{Z}^2} \alpha_{k_1,k_2}\phi_{j,k_1,k_2}|^2 dxdy \geq$$

$\gamma \sum_{(k_1,k_2)\in C^n} |\alpha_{k_1+l_1,k_2+l_2}|^2 =$
$\gamma \sum_{(k_1,k_2)\in C^{\eta}_{j,l_1,l_2}} |\alpha_{k_1,k_2}|^2$
and then

$$\int\int_{\Delta} |\sum_{(k_1,k_2)\in\mathbb{Z}^2} \alpha_{k_1,k_2}\phi_{j,k_1,k_2}|^2 dxdy \geq \gamma \sum_{T^{\eta}_{j,l_1,l_2}\subset\Delta} \sum_{(k_1,k_2)\in C^{\eta}_{j,l_1,l_2}} |\alpha_{k_1,k_2}|^2$$
$$\geq \gamma \sum_{(k_1,k_2)\in C_j} |\alpha_{k_1,k_2}|^2. \quad \blacksquare$$

We will make crucial use of the projection operators on the scaling spaces. In fact, combining Lemma 4.1 with Lemma 4.2 yields the following result which is very important for regularity criterion.

Theorem 4.1 *Let* $\Delta = \{(x,y) \in [-1,1]\times[0,1],\ y \leq 1-|x|\}$ *and* $j_0 \in \mathbb{N}$ *such that* $2^{j_0} \geq 4N-4$. *Let* $(V_j(\mathbb{R}^2))_{j\in\mathbb{Z}}$ *be a regular multiresolution analysis of* $L^2(\mathbb{R}^2)$. *We assume that there exists a projection operator* A_j *onto* $V_j(\mathbb{R}^2)$ *such that*

i) $A_{j+1}oA_j = A_joA_j = A_j$

ii) $\|F\|^2_{H^s(\mathbb{R}^2)} \approx \|A_0 f\|^2_{L^2(\mathbb{R}^2)} + \sum_{j\geq 0} 2^{2js}\|A_{j+1}F - A_jF\|^2_{L^2(\mathbb{R}^2)}$.

If P_j *is a projection operator from* $L^2(\Delta)$ *onto* $V_j(\Delta)$ *such that, for a constant* β *and* $j \geq j_0$, P_j *satisfies:*

$$\|P_j f\|^2_{L^2(\Delta)} \leq \beta\|f\|^2_{L^2(\Delta)} \tag{4.3}$$

then, we have

$$\forall f \in H^s(\Delta),\ \|f\|^2_{H^s(\ \Delta\ )} \approx \|P_{j_0} f\|^2_{L^2(\Delta)} + \sum_{j\geq j_0} 2^{2js}\|P_{j+1}f - P_j f\|^2_{L^2(\Delta)}. \tag{4.4}$$

Proof. We write $f = F_{/\Delta}$ and $F = (F - A_jF) + A_jF$. We have

$$\begin{aligned}\|P_{j+1}f - P_jf\|^2_{L^2(\Delta)} &= \|(P_{j+1}-P_j)(F-A_jF)_{/\Delta}\|^2_{L^2(\Delta)}\\ &\leq \beta\|F-A_jF\|^2_{L^2(\Delta)}\end{aligned}$$

where β is a positive constant independent of j. Then, we have

$$\begin{aligned}
\sum_{j\geq j_0} 2^{2js}\|P_{j+1}f-P_jf\|^2_{L^2(\Delta)} &\leq \beta\sum_{j\geq j_0} 2^{2js}\|F-A_jF\|^2_{L^2(\Delta)}\\
&\leq \beta\sum_{j\geq j_0} 2^{2js}\|F-A_jF\|^2_{L^2(\mathbb{R}^2)}\\
&\leq \beta\sum_{j\geq j_0} 2^{2js}\|\sum_{p\geq j+1}(A_p-A_{p-1})F\|^2_{L^2(\mathbb{R}^2)}\\
&\leq \beta\sum_{j\geq j_0}\{\sum_{p\geq j+1} 2^{(j-p)s}2^{2ps}\|(A_p-A_{p-1})F\|^2_{L^2(\mathbb{R}^2)}\}^2.
\end{aligned}$$

It's a convolution $\ell^1 o\ell^2\subset\ell^2$, then we get the first inequality. To prove the reverse inequality, we write $f=F_{/\Delta}$ and $F=E_0(P_0f)+\sum_{j\geq 0}E_{j+1}(P_{j+1}f-P_jf)$ where E_j is the extension operator described in Definition 4.1. Then, we have:

$$\|f\|^2_{H^s(\Delta)}\leq\|F\|^2_{H^s(\mathbb{R}^2)}\approx\|A_0f\|^2_{L^2(\mathbb{R}^2)}+\sum_{j\geq 0}2^{2js}\|A_{j+1}F-A_jF\|^2_{L^2(\mathbb{R}^2)}$$

and

$$A_{j+1}F-A_jF=\sum_{l\geq j}(A_{j+1}-A_j)E_{l+1}(P_{l+1}f-P_lf).$$

Then, we get for a constant M:

$$\begin{aligned}
2^{2js}\|A_{j+1}F-A_jF\|^2_{L^2(\mathbb{R}^2)} &\leq \sum_{l\geq j}2^{js}\|(A_{j+1}-A_j)E_{l+1}(P_{l+1}f-P_lf)\|^2_2\\
&\leq \sum_{l\geq j}M\alpha\|P_{l+1}f-P_lf\|^2_{L^2(\Delta)}2^{ls}2^{(j-l)s}.
\end{aligned}$$

It's a convolution $\ell^2 o\ell^1\subset\ell^2$. The same remarks as above give the result. Thus, Theorem 4.1 is proved. ■

Lemma 4.2 and Theorem 4.1 are the bases for our strategy: in order to get tool for regularity analysis of functions defined on the triangle, we shall try to define nice equicontinuous families of projection operators on the spaces $V_j(\Delta)$.

5 The spaces $w_j(\Delta)$ and $W_j(\Delta)$

We have $\dim v_j(\Delta)=2^{2j}-(6N-5)2^j+9N^2-15N+6$. Then, we get that $\dim w_j(\Delta)=3\times 2^{2j}-(6N-5)2^j$. We denote by

$K_j = \{(k_1, k_2) \in \mathbb{Z}^2, -2^j+2N-1 \leq k_1 \leq -N$ and $0 \leq k_2 \leq k_1+2^j-2N+1$ or $-N+1 \leq k_1 \leq 2^j-4N+2$ and $0 \leq k_2 \leq -k_1+2^j-4N+2\}$ (5.1)

and

$$\begin{aligned} L_{j+1} &= \{(2k_1, 2k_2), (k_1, k_2) \in K_j\} \cup \{(2k_1+2N-1, 2k_2), (k_1, k_2) \in K_j\} \\ &\cup \{(2k_1, 2k_2+2N-1), (k_1, k_2) \in K_j\} \\ &\cup \{(2k_1+2N-1, 2k_2+2N-1), (k_1, k_2) \in K_j\}. \end{aligned} \tag{5.2}$$

Observe that we can expect a simple description of the space $v_j(\Delta)$ which will be expressed by $v_j(\Delta) = vect\{\varphi_{j,k_1} \otimes \varphi_{j,k_2},\ (k_1, k_2) \in K_j\}$. Let $x_j(\Delta)$ be a supplement of $v_j(\Delta)$ into $v_{j+1}(\Delta)$, then $x_j(\Delta)$ has the following basis: $\varphi_{j,k_1} \otimes \psi_{j,k_2}$ with $(k_1, k_2) \in K_j$; $\psi_{j,k_1} \otimes \varphi_{j,k_2}$ with $(k_1, k_2) \in K_j$; $\psi_{j,k_1} \otimes \psi_{j,k_2}$ with $(k_1, k_2) \in K_j$ and $\varphi_{j+1,k_1} \otimes \varphi_{j+1,k_2}$ with $(k_1, k_2) \in K_{j+1} \setminus L_{j+1}$. We have exactly $3 \times 2^{2j} - (6N-5)2^j$ functions which are linearly independent. To prove this result, it is enough to take their restrictions to every square in Δ and apply Proposition 2.1. We split the family into the following sets:

i) interior wavelets: $\varphi_{j,k_1} \otimes \psi_{j,k_2}$, $\psi_{j,k_1} \otimes \varphi_{j,k_2}$ and $\psi_{j,k_1} \otimes \psi_{j,k_2}$ with $-2^j+2N-1 \leq k_1 \leq -N$ and $0 \leq k_2 \leq k_1+2^j-2N+1$ or $-N+1 \leq k_1 \leq 2^j-4N+2$ and $0 \leq k_2 \leq -k_1+2^j-4N+2$

ii) interior corner wavelets: $\varphi_{j+1,k_1} \otimes \varphi_{j+1,k_2}$ with $(k_1, k_2) \in K_{j+1} \setminus L_{j+1}$.

Orthonormalization of the interior corner wavelets gives an orthonormal basis ϖ_{j,k_1,k_2} for $w_j(\Delta)$ where $(k_1, k_2) \in K_{j+1} \setminus K_j$. Thus, We define the orthogonal projection operator Q_j^0 onto $w_j(\Delta)$ as

$$Q_j^0 f = \sum_{(k_1,k_2) \in K_{j+1} \setminus K_j} < f / \varpi_{j,k_1,k_2} > \varpi_{j,k_1,k_2}. \tag{5.3}$$

We can now establish the first main result of this section.

Theorem 5.1 *Let* $2^{j_0} \geq 4N-4$. *Then:*

a) for $f \in L^2(\Delta)$, *we have* $\|f\|^2_{L^2(\Delta)} = \|P^0_{j_0} f\|^2_{L^2(\Delta)} + \sum_{j=j_0}^{j=\infty} \|Q^0_j f\|^2_{L^2(\Delta)}$,

b) for $f \in H^s_0(\Delta)$, *we have* $\|f\|^2_{H^s(\Delta)} \approx \|P^0_{j_0} f\|^2_{L^2(\Delta)} + \sum_{j=j_0}^{j=\infty} 4^{sj} \|Q^0_j f\|^2_{L^2(\Delta)}$.

Proof. a) is a classical result in wavelet theory.
b) follows from Lemma 4.2 and Theorem 4.1. ■

We study now the space $W_j(\Delta)$. The construction of wavelets here is more complicated due to boundary conditions and the specific geometry of the triangle. We have $\dim V_j(\Delta) = 2^{2j} + (6N-5)2^j + (2N-2)^2$. Then we get that $\dim W_j(\Delta) = 3 \times 2^{2j} + (6N-5)2^j$. Let $X_j(\Delta)$ be a supplement of $V_j(\Delta)$ into $V_{j+1}(\Delta)$, then $X_j(\Delta)$ contains the following functions: $\varphi_{j,k_1} \otimes \psi_{j,k_2}$ with $(k_1,k_2) \in K_j$; $\psi_{j,k_1} \otimes \varphi_{j,k_2}$ with $(k_1,k_2) \in K_j$ and $\psi_{j,k_1} \otimes \psi_{j,k_2}$ with $(k_1,k_2) \in K_j$. We have exactly $3(2^{2j} - (6N-5)2^j + 9N^2 - 15N + 6)$ (interior wavelets) functions which are linearly independent. We must add $4(6N-5)2^j - 3(9N^2 - 15N + 6)$ functions (edge wavelets). It is very important to observe that the tensorization of Meyer's Lemma (Lemma 2.2) gives a basis for the supplement of V_j into V_{j+1} in the cases of a cube or a L-shaped domain (see [10] and [14]). But, it gives in our case only a generating system of $V_{j+1}(\Delta)$ which is not linearly independent. In fact, let us consider the following sets:

i) the family $\varphi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ with $k_1 = -2^j - 2N + 2 + p$, $N-1 \le p \le 2^j - 2$ and $-N+1 \le k_2 \le p - N + 1$; the family $\varphi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ with $k_1 = -2N + 1 + p$, $0 \le p \le 2N-1$ and $-N+1 \le k_2 \le 2^j - N$ and the family $\varphi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ with $k_1 = 2 - N + p$, $N-1 \le p \le 2^j - 2$ and $-N+1 \le k_2 \le -p + 2^j - 2$

ii) the family $\psi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $k_1 = -2^j - 2N + 2 + p$, $N-1 \le p \le 2^j - 2$ and $-2N+2 \le k_2 \le p$; the family $\psi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $k_1 = -2N + 1 + p$, $0 \le p \le 2N-1$ and $-2N+2 \le k_2 \le 2^j - 1$ and the family $\psi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $k_1 = -N + 2 + p$, $N-1 \le p \le 2^j - 2$ and $-2N+2 \le k_2 \le -p + 2^j + N - 3$

iii) the family $\psi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ with $k_1 = -2^j - 2N + 2 + p$, $N-1 \le p \le 2^j - 2$ and $-N+1 \le k_2 \le p - N + 1$; the family $\psi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ with $k_1 = -2N + 1 + p$, $0 \le p \le 2N-1$ and $-N+1 \le k_2 \le 2^j - N$ and the family $\psi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ with $k_1 = -N + 2 + p$, $N-1 \le p \le 2^j - 2$ and $-N+1 \le k_2 \le -p + 2^j - 2$.

Thus, we get $(3.2^{2j} + (10N-7)\,2^j - 3N^2 + 3N)$ functions which are more than $\dim W_j(\Delta)$ (some of these functions are in $V_j(\Delta)$). Then, by using incomplete basis theorem, we get the following result.

Theorem 5.2 *Let* $2^{j_0} \ge 4N - 4$. *Then:*

a) there exist $(3 \times 2^{2j} + (6N-5)2^j)$ *functions* Ψ_{j,k_1,k_2} *such that the functions* $\phi_{j,k_1,k_2/\Delta}$ *for* $V_j(\Delta)$ *where* $(k_1,k_2) \in M_j$ *and* Ψ_{j,k_1,k_2} *where* $(k_1,k_2) \in M_{j+1} \setminus M_j$, *form an orthonormal basis for* $V_{j+1}(\Delta)$, *b) for* $f \in L^2(\Delta)$, *we have* $\|f\|^2_{L^2(\Delta)} = \|P_{j_0} f\|^2_{L^2(\Delta)} + \sum_{j=j_0}^{j=\infty} \|Q_j f\|^2_{L^2(\Delta)}$,

c) for $f \in H^s(\Delta)$, *we have* $\|f\|^2_{H^s(\Delta)} \approx \|P_{j_0} f\|^2_{L^2(\Delta)} + \sum_{j=j_0}^{j=\infty} 4^{sj} \|Q_j f\|^2_{L^2(\Delta)}$.

Proof. a) We consider interior wavelets $\varphi_{j,k_1} \otimes \psi_{j,k_2}$, $\psi_{j,k_1} \otimes \varphi_{j,k_2}$ and $\psi_{j,k_1} \otimes \psi_{j,k_2}$ with $-2^j + 2N - 1 \leq k_1 \leq -N$ and $0 \leq k_2 \leq k_1 + 2^j - 2N + 1$ or $-N + 1 \leq k_1 \leq 2^j - 4N + 2$ and $0 \leq k_2 \leq -k_1 + 2^j - 4N + 2$, and we complete this system from the collection described above (edge and corner wavelets). Next, we apply Gram-Schmidt to edge and corner wavelets.
b) is a classical result in wavelet theory.
c) follows from Lemma 4.2 and Theorem 4.1. ∎

To illustrate Theorem 5.2, we study two particular cases ($N = 1$ and $N = 2$). We consider the Haar basis (which corresponds to the case $N = 1$). Proposition 3.2 shows that $V_j(\Delta)$ has the following basis: the family $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $k_1 = -2^j + p$, $0 \leq p \leq 2^j - 1$ and $0 \leq k_2 \leq p$ and the family $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $k_1 = p$, $0 \leq p \leq 2^j - 1$ and $0 \leq k_2 \leq 2^j - 1 - p$. We can split these families into the following sets:

i) interior functions: $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $-2^j + 1 \leq k_1 \leq -1$ and $0 \leq k_2 \leq k_1 + 2^j - 1$; $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $0 \leq k_1 \leq 2^j - 2$ and $0 \leq k_2 \leq -k_1 + 2^j - 2$

ii) edge functions: $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $-2^j + 1 \leq k_1 \leq -2$ and $k_2 = k_1 + 2^j$; $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $1 \leq k_1 \leq 2^j - 2$ and $k_2 = 2^j - 1 - k_1$

iii) exterior corner functions: $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $k_1 = -2^j$ and $k_2 = 0$; $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $k_1 = 2^j - 1$ and $k_2 = 0$

iv) interior corner functions: $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $-1 \leq k_1 \leq 0$ and $k_2 = 2^j - 1$.

Orthonormalization of the border functions (edge and corner functions) gives an orthonormal basis for $V_j(\Delta)$ where $\dim V_j(\Delta) = 2^{2j} + 2^j$. We study now the space $W_j(\Delta)$. The construction of wavelets here is more simple due to small support of the Haar basis. We have $\dim W_j(\Delta) = 3 \times 2^{2j} + 2^j$. Let $X_j(\Delta)$ be a supplement of $V_j(\Delta)$ into $V_{j+1}(\Delta)$, then $X_j(\Delta)$ has the following Riesz basis:

i) the family $\varphi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ with $k_1 = -2^j + p$, $0 \leq k_2 \leq p$ and $0 \leq p \leq 2^j - 1$ and the family $\varphi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ with $k_1 = p$, $0 \leq k_2 \leq 2^j - p - 1$ and $0 \leq p \leq 2^j - 1$

ii) the family $\psi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $k_1 = -2^j + p$, $0 \leq k_2 \leq p$ and $0 \leq p \leq 2^j - 1$ and the family $\psi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $k_1 = p$, $0 \leq k_2 \leq 2^j - 1 - p$ and $0 \leq p \leq 2^j - 1$

iii) the family $\psi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ with $k_1 = -2^j + p$, $0 \leq k_2 \leq p - 1$ and the family $\psi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ with $k_1 = p - 1$, $0 \leq k_2 \leq 2^j - p - 1$ and $1 \leq p \leq 2^j - 1$.

We have exactly $(3 \times 2^{2j} + 2^j)$ functions which are linearly independent because the third collection has a support in the interior of Δ and the boundary functions are in the sets i) and ii).

We study now the case $N = 2$. Proposition 3.2 shows that $V_j(\Delta)$ has the following basis : the family $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $k_1 = -2^j + p$, $-2 \leq p \leq 2^j - 4$ and $-2 \leq k_2 \leq p + 2$, the family $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $-3 \leq k_1 \leq 0$ and $-2 \leq k_2 \leq 2^j - 1$ and the family $\varphi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $k_1 = p$, $1 \leq p \leq 2^j - 1$ and $-2 \leq k_2 \leq 2^j - 1 - p$. Orthonormalization of the border functions gives an orthonormal basis for $V_j(\Delta)$ where $\dim V_j(\Delta) = 2^{2j} + 7 \times 2^j + 4$. We describe now a basis of the associated space $W_j(\Delta)$. The construction of wavelets here is different from the case of the Haar basis ($N = 1$). We have $\dim W_j(\Delta) = 3 \times 2^{2j} + 7 \times 2^j$. Let $X_j(\Delta)$ be a supplement of $V_j(\Delta)$ into $V_{j+1}(\Delta)$, then $X_j(\Delta)$ has the following Riesz basis:

i) the family $\varphi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ with $k_1 = -2^j + p$, $-1 \leq k_2 \leq p + 1$ and $-1 \leq p \leq 2^j - 4$, the family $\varphi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ with $-3 \leq k_1 \leq 0$ and $-1 \leq k_2 \leq 2^j - 2$ and the family $\varphi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ with $k_1 = p$, $-1 \leq k_2 \leq 2^j - p - 2$ and $1 \leq p \leq 2^j - 2$

ii) the family $\psi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $k_1 = -2^j + p$, $-2 \leq k_2 \leq p + 2$ and $-1 \leq p \leq 2^j - 4$, the family $\psi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $-3 \leq k_1 \leq 0$ and $-2 \leq k_2 \leq 2^j - 1$ and the family $\psi_{j,k_1} \otimes \varphi_{j,k_2/\Delta}$ with $k_1 = p$, $-2 \leq k_2 \leq 2^j - 1 - p$ and $1 \leq p \leq 2^j - 2$

iii) the family $\psi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ with $k_1 = -2^j + p$, $0 \leq k_2 \leq p - 1$ and $1 \leq p \leq 2^j - 4$, the family $\psi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ with $-3 \leq k_1 \leq 0$ and $0 \leq k_2 \leq 2^j - 4$, the family $\psi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ with $k_1 = p$, $0 \leq k_2 \leq 2^j - 4 - p$ and $1 \leq p \leq 2^j - 4$ and the family $\psi_{j,k_1} \otimes \psi_{j,k_2/\Delta}$ where $(k_1, k_2) \in \{(-2^j + 1, -1), (-2, 2^j - 3), (-1, 2^j - 3), (2^j - 4, -1)\}$.

We have exactly $(3 \times 2^{2j} + 7 \times 2^j)$ functions which are linearly independent due to Proposition 2.1.

Remark 5.1 *The cases $N = 1$ and $N = 2$ illustrate clearly Theorem 5.2. The general idea consists to take near wavelets which satisfy Proposition 2.1 or Lemma 2.2.*

Remark 5.2 *Starting from the OMRA of I. Daubechies and using the method of "integration and derivation" introduced by P.G. Lemarié in [16], we can describe in the same way biorthogonal wavelet bases on the triangle.*

6 Conclusion

We have constructed in this paper two elementary multiresolution analyses on the triangle Δ. In the first one, we used a direct method based on Proposition 2.1 to define an orthonormal multiresolution analysis on Δ which is adapted to scales to provide orthogonal wavelet bases with compact support in Δ. This analysis is adapted for the study of the Sobolev spaces $H_0^s(\Delta)$ ($s \in \mathbb{N}$). In the second one, we construct an orthonormal multiresolution analysis on Δ satisfying boundary conditions. These bases are associated to simple algorithms and are adapted for the study of regular spaces on Δ. Lemma 4.2 and Theorem 4.1 permit to get the norm equivalences in the two cases.

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