Abstract

In this work, we revisit some well-known results in the theory of initial and progressive enlargement of a reference filtration $\mathcal{F}$ with a random time $\tau$, providing, under an equivalence assumption slightly stronger than the absolute continuity assumption of Jacod, alternative proofs to the already existing ones. We also characterize martingales in the enlarged filtrations in terms of martingales in the reference filtration, and we study representation theorems in the enlarged filtrations.

Keywords: initial and progressive enlargement of filtration, predictable projection, canonical decomposition, predictable representation theorem.

1 Introduction and preliminaries

We consider the case where a filtration $\mathcal{F}$ is enlarged to give a filtration $\tilde{\mathcal{F}}$, by means of a finite positive random variable $\tau$. In the literature, there are two ways to realize such an enlargement: either all of a sudden at time 0 (initial enlargement), or progressively, by considering the smallest $\sigma$-algebra containing $\mathcal{F}$ that makes $\tau$ a stopping time (progressive enlargement).

The “pioneers” who started exploring this research field, at the end of the seventies, were Barlow [4], Jacod, Jeulin and Yor (see the references that follow in the text). The main questions that raised were the following: “Does any $\mathcal{F}$-martingale $X$ remain an $\tilde{\mathcal{F}}$ semi-martingale?” And, if it does: “What is the semi-martingale decomposition in $\tilde{\mathcal{F}}$ of the $\mathcal{F}$-martingale $X$?”
The main contribution of the present work is to show how, under a specific equivalence assumption, stronger than Jacod’s one in [15], some well-known fundamental results can be proved in an alternative (and, in some cases, simpler) way. We make precise that the goal of this paper is not to present the results in the most general case, neither to study the needed and difficult regularity properties, for which we refer to existing papers.

Let us start by motivating the title, by introducing some notation and by stating the preliminary results that are needed henceforth. Inspired by a visit to the Tunisian archaeological site of Carthage, where one can find remains of THREE levels of different civilizations, we decided to use the catchy adjective “Carthaginian” associated with filtration, since in this paper there will be THREE levels of filtrations.

We consider, then, three nested filtrations
\[ \mathcal{F} \subset \mathcal{G} \subset \mathcal{G}^\tau, \]
where \( \mathcal{G} \) and \( \mathcal{G}^\tau \) stand, respectively, for the progressive and the initial enlargement of \( \mathcal{F} \) with a finite random time \( \tau \) (i.e., a finite non-negative random variable).

Under a specific assumption (see the (\( \mathcal{E} \))-Hypothesis below), we address the following problems:

- Characterization of \( \mathcal{G} \)-martingales and \( \mathcal{G}^\tau \)-martingales in terms of \( \mathcal{F} \)-martingales (in Section 2);
- Canonical decomposition of an \( \mathcal{F} \)-martingale, as a semimartingale, in \( \mathcal{G} \) and \( \mathcal{G}^\tau \) (in Section 3);
- Predictable Representation Theorem in \( \mathcal{G} \) and \( \mathcal{G}^\tau \) (in Section 4).

The exploited idea is the following: assuming that the \( \mathcal{F} \)-conditional law of \( \tau \) is equivalent to the law of \( \tau \), after an ad hoc change of probability measure, the problem reduces to the case where \( \tau \) and \( \mathcal{F} \) are independent. Under this newly introduced probability measure, working in the initially enlarged filtration is “easy”. Then, under the original probability measure, for the initially enlarged filtration, the results are achieved by means of Girsanov’s theorem. Finally, by projection, one obtains the results of interest in the progressively enlarged filtration (notice that, alternatively, they can be obtained with another application of Girsanov’s theorem, starting from the newly introduced probability measure, with respect to the progressively enlarged filtration).

The “change of probability measure” viewpoint for treating problems on enlargement of filtrations was remarked in the early 80’s and developed by Song [21] (see also Jacod [15], Section 5). For what concerns the idea of recovering the results in the progressively enlarged filtration starting from the ones in the initially enlarged one, we have to cite Yor [24].

Let us now become more precise about the setup and the preliminary results. In all the paper, we consider a probability space \( (\Omega, \mathcal{A}, P) \) equipped with a filtration \( \mathcal{F} = (\mathcal{F}_t)_{t \geq 0} \) satisfying the usual hypotheses of right-continuity and completeness, and where \( \mathcal{F}_0 \) is the trivial \( \sigma \)-field. Let \( \tau \) be a finite random time with law \( \nu, \nu(du) = P(\tau \in du) \). We assume that \( \nu \) has no atoms.

We denote by \( \mathcal{P}(\mathcal{F}) \) (resp. \( \mathcal{O}(\mathcal{F}) \)) the predictable (resp. optional) \( \sigma \)-algebra corresponding to \( \mathcal{F} \) on \( \mathbb{R}^+ \times \Omega \).

Our standing assumption is the following:

**Assumption 1.1 (\( \mathcal{E} \))-Hypothesis**

The \( \mathcal{F} \)-(regular) conditional law of \( \tau \) is equivalent to the law of \( \tau \). Namely,

\[ P(\tau \in du|\mathcal{F}_t) \sim \nu(du) \quad \text{for every } t \geq 0, \, \mathbb{P} - a.s. \]
Notice that this assumption, in the case when \( t \in [0, T] \), corresponds to the equivalence assumption in Föllmer et Imkeller [12], Amendinger’s thesis [1, Assumption 0.2] and to hypothesis (HJ) in the papers from Grorud and Pontier (see, e.g., [13]).

Amongst the consequences of the (E)-Hypothesis, the existence and regularity of the conditional density, for which we refer to Amendinger’s reformulation (see remarks, page 17 of [1]) of Jacod’s result (Lemma 1.8 in [15]): there exists a strictly positive \( \mathcal{O}(\mathcal{F}) \otimes \mathcal{B}(\mathbb{R}^+) \)-measurable function \( (t, \omega, u) \rightarrow p_t(\omega, u) \), such that for every \( u \in \mathbb{R}^+ \), \( p(u) \) is a càdlàg \((\mathbb{P}, \mathcal{F})\)-martingale and

\[
\mathbb{P}(\tau > \theta | \mathcal{F}_t) = \int_0^\infty p_t(u)\nu(du) \quad \text{for every } t \geq 0, \quad \mathbb{P} - \text{a.s.}
\]

In particular, \( p_0(u) = 1 \) for every \( u \in \mathbb{R}^+ \). This process \( p \) is called the \((\mathbb{P}, \mathcal{F})\)-conditional density of \( \tau \) with respect to \( \nu \), or the density of \( \tau \) if there is no ambiguity.

Furthermore, under the (E)-Hypothesis, the assumption that \( \nu \) has no atoms implies that the default time \( \tau \) avoids the \( \mathbb{F}\)-stopping times, i.e., \( \mathbb{P}(\tau = \xi) = 0 \) for every \( \mathbb{F}\)-stopping time \( \xi \) (see, e.g., Corollary 2.2 in El Karoui et al. [11]).

The initial enlargement of \( \mathbb{F} \) with \( \tau \), denoted by \( \mathbb{G}^\tau = (\mathcal{G}_t^\tau, t \geq 0) \) is defined as \( \mathcal{G}_t^\tau = \mathcal{F}_t \vee \sigma(\tau) \). It was shown in Amendinger [1] (cf. Proposition 1.10 therein) that the strict positiveness of \( p \) implies the right-continuity of the filtration \( \mathcal{G}^\tau \).

Let \( \mathcal{H} = (\mathcal{H}_t)_{t \geq 0} \) denote the smallest filtration with respect to which \( \tau \) is a stopping time, i.e., \( \mathcal{H}_t = \sigma(1_{\tau \leq s}, s \leq t) \). This filtration is continuous on right (since the law of \( \tau \) has no atoms). The progressive enlargement of \( \mathbb{F} \) with the random time \( \tau \), denoted by \( \mathbb{G} = (\mathcal{G}_t)_{t \geq 0} \), is defined as the right-continuous regularization of \( \mathbb{F} \vee \mathcal{H} \).

Now, we consider the change of probability measure introduced, independently, by Grorud and Pontier in [13] and by Amendinger in [1]. Having verified that the process \( L \), defined by \( L_t = \frac{1}{p_t(\tau)} \), \( t \geq 0 \), is a \((\mathbb{P}, \mathcal{G}^\tau)\)-martingale, with \( \mathbb{E}(L_1) = L_0 = 1 \), these authors defined a locally equivalent probability measure \( \mathbb{P}^* \) setting

\[
d\mathbb{P}^*|\mathcal{G}_t = L_t d\mathbb{P}|\mathcal{G}_t = \frac{1}{p_t(\tau)} d\mathbb{P}|\mathcal{G}_t^\tau.
\]

They proved that, under \( \mathbb{P}^* \), the random time \( \tau \) is independent of \( \mathcal{F}_t \) for any \( t \geq 0 \) and, moreover, that

\[
\mathbb{P}^*|\mathcal{F}_t = \mathbb{P}|\mathcal{F}_t \text{ for any } t \geq 0, \quad \mathbb{P}^*|\sigma(\tau) = \mathbb{P}|\sigma(\tau).
\]

Note that the above properties imply that \( \mathbb{P}^*(\tau \in du|\mathcal{F}_t) = \mathbb{P}^*(\tau \in du) \), so that the \((\mathbb{P}^*, \mathbb{F})\)-density of \( \tau \), denoted by \( p^*(u) \), \( u \geq 0 \), is a constant equal to one, \( \mathbb{P}^* \otimes \nu \)-a.s.

**Remark 1.1** It is not possible to state that: “Under \( \mathbb{P}^* \), the r.v. \( \tau \) is independent of \( \mathcal{F}_\infty \)”, since we do not know a priori whether \( p(\tau) \) is uniformly integrable or not, so that \( \mathcal{P}^* \) is not even defined on \( \mathcal{G}_\infty^\tau \). A similar problem is studied by Föllmer and Imkeller in [12] (it is therein called “paradox”) in the case where the reference (canonical) filtration is enlarged by the information about the endpoint at time \( t = 1 \). In our setting, it corresponds to the case where \( \tau \in \mathcal{F}_\infty \) and \( \tau \notin \mathcal{F}_t, \forall t \).

**Remark 1.2** Let \( x = (x_t, t \geq 0) \) be a \((\mathbb{P}, \mathbb{F})\)-martingale. Since \( \mathbb{P} \) and \( \mathbb{P}^* \) coincide on \( \mathcal{F} \), \( x \) is a \((\mathbb{P}^*, \mathcal{F})\)-martingale, hence, using the fact that \( \tau \) is independent of \( \mathbb{F} \) under \( \mathbb{P}^* \), a \((\mathbb{P}^*, \mathcal{G})\)-martingale (and also a \((\mathbb{P}^*, \mathcal{G}^\tau)\)-martingale). Because of these facts, the measure \( \mathbb{P}^* \) is called by Amendinger “martingale preserving probability measure under initial enlargement of filtrations”.

Notation 1.1 In this paper, as we mentioned, we deal with three different levels of information and two equivalent probability measures. In order to distinguish objects defined under \( P \) and under \( P^* \), we will use a superscript \( * \) when working under \( P^* \). For example, \( \mathbb{E} \) and \( \mathbb{E}^* \) stand for the expectations under \( P \) and \( P^* \), respectively. For what concerns the filtrations, when necessary, we will use the following illustrating notation: \( x, X, X^\tau \) to denote processes adapted to \( F \), \( G \) and \( G^\tau \), respectively. (We shall not use the same notation for processes stopped at \( \tau \), so that there is no possible confusion for the notation \( X^\tau \).)

We conclude this introductory section with the following general result, that will be very useful in the sequel.

Proposition 1.1 Projection
Let \( \tilde{F} \) be a filtration larger than \( F \), i.e., \( F \subset \tilde{F} \). If \( x \) is a uniformly integrable \( F \)-martingale, then there exists a \( \tilde{F} \)-martingale \( \tilde{x} \), such that \( \mathbb{E}(\tilde{x}_t | F_t) = x_t \), \( t \geq 0 \).

Proof. The process \( \tilde{x} \) defined by \( \tilde{x}_t := \mathbb{E}(x_\infty | \tilde{F}_t) \) is an \( \tilde{F} \)-martingale, and \( \mathbb{E}(\tilde{x}_t | F_t) = \mathbb{E}(\mathbb{E}(x_\infty | \tilde{F}_t) | F_t) = \mathbb{E}(x_\infty | F_t) = x_t \).

Remark 1.3 The uniqueness of such a martingale \( \tilde{x} \) is not claimed in the above proposition and is not true in general.

1.1 Characterization of different measurability properties

Before focusing on the three topics announced since the beginning, we recall some important results on the characterization of \( G^\tau \) or \( G^\tau \)-measurable random variables and \( G^\tau \) or \( G \)-predictable processes. The necessary part of the result below, in the case of predictable processes, is due to Jeulin [17, Lemma 3.13]. See also Yor [24].

Proposition 1.2 One has
(i) A random variable \( Y^\tau \) is \( G^\tau \)-measurable if and only if it is of the form \( Y^\tau_t(\omega) = y_t(\omega, \tau(\omega)) \) for some \( F_t \otimes B(\mathbb{R}^+) \)-measurable random variable \( y_t(\cdot, u) \).

(ii) A process \( Y^\tau \) is \( G^\tau \)-predictable if and only if it is of the form \( Y^\tau_t(\omega) = y_t(\omega, \tau(\omega)), t \geq 0 \), where \( (t, \omega, u) \mapsto y_t(\omega, u) \) is a \( \mathcal{P}(F) \otimes B(\mathbb{R}^+) \)-measurable function.

Proof. The proof of part (i) is based on the fact that \( G^\tau \)-measurable random variables are generated by random variables of the form \( X_t(\omega) = x_t(\omega)f(\tau(\omega)) \), with \( x_t \in F_t \) and \( f \) bounded Borel on \( \mathbb{R}^+ \).

(ii) For the necessity, it suffices to notice that processes of the form \( X_t := x_t f(\tau), t \geq 0 \), where \( x \) is \( F \)-predictable and \( f \) is a bounded Borel function on \( \mathbb{R}^+ \), generate the \( G^\tau \)-predictable \( \sigma \)-field. We then obtain the result applying a monotone class argument.

The converse is obvious.

For what concerns the progressive enlargement setting, the following result is analogous to Proposition 1.2. The necessity of part (ii) is already proved in Jeulin [17, Lemma 4.4].
Proposition 1.3 One has

(i) A random variable \( Y_t \) is \( G_t \)-measurable if and only if it is of the form \( Y_t(\omega) = \tilde{y}_t(\omega)1_{t \leq \tau(\omega)} + \tilde{y}_t(\omega, \tau(\omega))1_{\tau(\omega) < t} \) for some \( F_t \)-measurable random variable \( \tilde{y}_t \) and some family of \( F_t \otimes \mathcal{B}(\mathbb{R}^+) \)-measurable random variables \( \tilde{y}_t(\cdot, u), t \geq u \).

(ii) A process \( Y \) is \( G \)-predictable if and only if it is of the form \( Y_t(\omega) = \tilde{y}_t(\omega)1_{t \leq \tau(\omega)} + \tilde{y}_t(\omega, \tau(\omega))1_{\tau(\omega) < t} \), \( t \geq 0 \), where \( \tilde{y} \) is \( F \)-predictable and \((t, \omega, u) \mapsto \tilde{y}_t(\omega, u)\) is a \( \mathcal{P}(\mathcal{F}) \otimes \mathcal{B}(\mathbb{R}^+) \)-measurable function.

Proof. For part (i) it suffices to recall that \( G_t \)-measurable random variables are generated by random variables of the form \( X_t(\omega) = x_t(\omega)f(t \wedge \tau(\omega)) \), with \( x_t \in \mathcal{F}_t \) and \( f \) a measurable bounded function on \( \mathbb{R}^+ \).

(ii) As previously done in the proof of Proposition 1.2, it suffices to notice that \( G \)-predictable processes are generated by processes of the form \( X_t = x_t1_{t \leq \tau} + \tilde{x}_tf(\tau)1_{\tau < t}, \ t \geq 0 \), where \( x, \tilde{x} \) are \( F \)-predictable and \( f \) is a measurable bounded function, defined on \( \mathbb{R}^+ \).

\( \Box \)

1.2 Expectation and projection tools

Lemma 1.4 Let \( Y_t^{\tau} = y_t(\tau) \) be a \( G_t^{\tau} \)-measurable random variable.

(i) If \( y_t(\tau) \) is \( \mathbb{P} \)-integrable and \( y_t(\tau) = 0 \) \( \mathbb{P} \)-a.s. then, for \( \nu \)-a.e. \( u \geq 0 \), \( y_t(u) = 0 \) \( \mathbb{P} \)-a.s.

(ii) For \( s \leq t \) one has, \( \mathbb{P} \)-a.s. (or equivalently \( \mathbb{P}^* \)-a.s.):

if \( y_t(\tau) \) is \( \mathbb{P}^* \)-integrable and if \( y_t(u) \) is \( \mathbb{P} \) (or \( \mathbb{P}^* \))-integrable for any \( u \)

\[
\mathbb{E}^*(y_t(\tau)|G_s^\tau) = \mathbb{E}^*(y_t(u)|F_s)|_{u=\tau} = \mathbb{E}(y_t(u)|F_s)|_{u=\tau}; \tag{1}
\]

if \( y_t(\tau) \) is \( \mathbb{P} \)-integrable

\[
\mathbb{E}(y_t(\tau)|G_s^\tau) = \frac{1}{p_s(\tau)} \mathbb{E}(y_t(u)p_t(u)|F_s)|_{u=\tau}. \tag{2}
\]

Proof. (i) We have, by applying Fubini-Tonelli’s Theorem,

\[
0 = \mathbb{E}(|y_t(\tau)|) = \mathbb{E}\left(\mathbb{E}(|y_t(\tau)||F_t)\right) = \mathbb{E}\left(\int_0^\infty |y_t(u)|p_t(u)\nu(du)\right).
\]

Then \( \int_0^\infty |y_t(u)|p_t(u)\nu(du) = 0 \) \( \mathbb{P} \)-a.s. and, given that \( p_t(u) \) is strictly positive for any \( u \) and that \( \nu \) is non atomic, we have that for \( \nu \)-almost every \( u, y_t(\cdot, u) = 0 \) \( \mathbb{P} \)-a.s.

(ii) The first equality in (1) is straightforward for elementary random variables of the form \( f(\tau)x_t \) from the independence between \( \tau \) and \( F_t \), for any \( t \geq 0 \). It is extended via the MCT (Monotone Class Theorem). The second equality follows from the fact that \( \mathbb{P} \) and \( \mathbb{P}^* \) coincide on \( F_t \), for any \( t \geq 0 \).

The result (2) is an immediate consequence of (1), since it suffices, by means of (conditional) Bayes’ formula, to pass under the measure \( \mathbb{P}^* \). Namely, for \( s < t \), we have

\[
\mathbb{E}(y_t(\tau)|G_s^\tau) = \frac{\mathbb{E}^*(y_t(\tau)p_t(\tau)|G_s^\tau)}{\mathbb{E}^*(p_t(\tau)|G_s^\tau)} = \frac{1}{p_s(\tau)} \mathbb{E}(y_t(u)p_t(u)|F_s)|_{u=\tau},
\]
where in the last equality we have used the previous result (1) and the fact that $p(\tau)$ is a $(\mathbb{P}^*, \mathcal{G}^\tau)$-martingale. Note that if $y_t(\tau)$ is $\mathbb{P}$-integrable, then $\mathbb{E}(\int_0^\infty |y_t(u)| p_t(u) \nu(du)) = \mathbb{E}(|y_t(\tau)|) < \infty$, which implies that $\mathbb{E}(|y_t(u)| p_t(u)) < \infty$.  

We will also need the following lemma.

**Lemma 1.5** Let $Y^\tau_t = y_t(\tau)$ be a $\mathcal{G}^\tau_t$-measurable, $\mathbb{P}$-integrable random variable. Then, for $s \leq t$,

$$\mathbb{E}(Y^\tau_t | \mathcal{G}_s) = \mathbb{E}(y_t(\tau) | \mathcal{G}_s) = \tilde{y}_s \mathbb{1}_{s < \tau} + \tilde{y}_s(\tau) \mathbb{1}_{s \leq \tau},$$

with

$$\tilde{y}_s = \frac{1}{G_s} \mathbb{E}\left( \int_s^{+\infty} y_t(u) p_t(u) \nu(du) | \mathcal{F}_s \right),$$

$$\tilde{y}_s(u) = \frac{1}{p_s(u)} \mathbb{E}(y_t(u)p_t(u) | \mathcal{F}_s).$$

**Proof.** From the above Proposition 1.3 it is clear that $\mathbb{E}(y_t(\tau) | \mathcal{G}_s)$ has to be written in the form $\tilde{y}_s \mathbb{1}_{s < \tau} + \tilde{y}_s(\tau) \mathbb{1}_{s \leq \tau}$. On the set $\{s < \tau\}$, we have, applying Lemma 3.1.2 in Bielecki et al. [6] and using the $(\mathcal{E})$-Hypothesis (see also El Karoui et al. [11] for analogous computations),

$$\mathbb{1}_{s < \tau} \mathbb{E}(y_t(\tau) | \mathcal{G}_s) = \mathbb{1}_{s < \tau} \frac{\mathbb{E}\mathbb{E}(y_t(\tau) \mathbb{1}_{s < \tau} | \mathcal{F}_t) | \mathcal{F}_s}{G_s} = \mathbb{1}_{s < \tau} \frac{1}{G_s} \mathbb{E}\left( \int_s^{+\infty} y_t(u) p_t(u) \nu(du) | \mathcal{F}_s \right) =: \mathbb{1}_{s < \tau} \tilde{y}_s.$$

On the complementary set we have, by applying Lemma 1.4,

$$\mathbb{1}_{s \leq \tau} \mathbb{E}(y_t(\tau) | \mathcal{G}_s) = \mathbb{1}_{s \leq \tau} \mathbb{E}(y_t(\tau) | \mathcal{G}^\tau_t | \mathcal{G}_s) = \mathbb{1}_{s \leq \tau} \frac{1}{p_s(\tau)} \mathbb{E}\left( y_t(u) p_t(u) | \mathcal{F}_s | \mathbb{1}_{u=\tau} \right) =: \mathbb{1}_{s \leq \tau} \tilde{y}_s(\tau).$$

For $s > t$, we obtain

$$\mathbb{E}(Y^\tau_t | \mathcal{G}_s) = \frac{1}{G_t} \int_t^{+\infty} y_t(u) p_t(u) \nu(du) \mathbb{1}_{s < \tau} + y_t(\tau) \mathbb{1}_{s \leq \tau}.$$  

Remark that, taking the right-continuous versions of various $\mathbb{P}$-martingales and the right-continuous version of the supermartingale $G$, we have obtained the right-continuous version of the $\mathbb{G}$-martingale $(\mathbb{E}(Y^\tau_t | \mathcal{G}_s), s \geq 0)$.  

As an application, projecting the martingale $L$ (defined earlier as $L_t = \frac{1}{p_t(\tau)}, t \geq 0)$ on $\mathbb{G}$ yields to the corresponding Radon-Nikodým density on $\mathbb{G}$:

$$d\mathbb{P}^*|\mathcal{G}_t = \ell_t d\mathbb{P}|\mathcal{G}_t,$$

with

$$\ell_t := \mathbb{E}(L_t | \mathcal{G}_t) = \mathbb{1}_{t < \tau} \frac{1}{G_t} \int_t^{+\infty} \nu(du) + \mathbb{1}_{t \leq \tau} \frac{1}{p_t(\tau)} = \mathbb{1}_{t < \tau} \frac{G(t)}{G_t} + \mathbb{1}_{t \leq \tau} \frac{1}{p_t(\tau)},$$

where we have used the notation $G$ (resp., $G(\cdot)$) for conditional survival process (resp., function) under the probability measure $\mathbb{P}$ (resp. $\mathbb{P}^*$).

More precisely,

$$G_t := \mathbb{P}(\tau > t | \mathcal{F}_t) = \int_t^{+\infty} p_t(u) \nu(du), \quad (3)$$

$$G(t) := \mathbb{P}^*(\tau > t | \mathcal{F}_t) = \mathbb{P}^*(\tau > t) = \mathbb{P}(\tau > t) = \int_t^{+\infty} \nu(du). \quad (4)$$
Note, in particular, that \((G_t)_{t \geq 0}\) is an \(F\) super-martingale, whereas \(G(\cdot)\) is a (deterministic) continuous and decreasing function. Furthermore, it is clear that, under the \((E)\)-Hypothesis, \(G\) and \(G(\cdot)\) do not vanish. Note also that \(\ell\) is indeed the right-continuous version of the \(G\)-martingale \((\mathbb{E}(L_t|G_t), t \geq 0)\).

We now recall some important facts concerning the compensated martingale of \(H\). We know, from the general theory (see, for example, El Karoui et al. [11]), that denoting by \(Y\) the default indicator process \(H_t = \mathbb{1}_{t \leq \tau}, t \geq 0\), the process \(M\) defined as

\[
M_t := H_t - \int_0^{t \wedge \tau} \lambda_s \nu(ds), \quad t \geq 0,
\]

with \(\lambda_t = \frac{\nu(t)}{\ell(t)}\), is a \((\mathbb{P}, G)\)-martingale and that

\[
M^*_t := H_t - \int_0^{t \wedge \tau} \lambda^*(s) \nu(ds), \quad t \geq 0,
\]

with \(\lambda^*(t) = \frac{1}{\ell'(t)}\), is a \((\mathbb{P}^*, G)\)-martingale. Furthermore, since \(\lambda^*\) is deterministic, \(M^*\) (being \(H\)-adapted) is a \((\mathbb{P}^*, \mathbb{H})\)-martingale, too.

We conclude this subsection with the following two propositions, concerning the predictable projection, respectively on \(F\) and on \(G\), of a \(G^\tau\)-predictable process. The first result is due to Jacod [15, Lemme 1.10].

**Proposition 1.6** Let \(Y^\tau = y(\tau)\) be a \(G^\tau\)-predictable, positive or bounded, process. Then, the predictable projection of \(Y^\tau\) on \(F\) is given by

\[
^{(p)}(Y^\tau)_t = \int_0^{\tau} y_t(u)\nu_{\tau-}(u)\nu(du).
\]

**Proof.** It is obtained by a monotone class argument and by using the definition of density of \(\tau\), writing, for “elementary” processes, \(Y^\tau_t := y_t f(\tau)\), with \(y\) a bounded \(\mathbb{F}\)-predictable process and \(f\) a bounded Borel function. For this, we refer to the proof of Lemma 1.10 in Jacod [15].

**Proposition 1.7** Let \(Y^\tau = y(\tau)\) be a \(G^\tau\)-predictable, positive or bounded, process. Then, the predictable projection of \(Y^\tau\) on \(G\) is given by

\[
^{(p)}(Y^\tau)_t = \frac{1}{G_t} \int_0^{\tau} y_t(u)\nu_{\tau-}(u)\nu(du) + \mathbb{1}_{\tau \leq \tau} y_t(\tau).
\]

**Proof.** In this proof, for clarity, the left-hand side superscript “\((p G)\)” denotes the predictable projection on \(G\), while the left-hand side superscript “\((p F)\)” indicates the predictable projection on \(F\). By the definition of predictable projection, we know (from Proposition 1.3 (ii)) that we are looking for a (unique) process of the form

\[
^{(p G)}(Y^\tau)_t = \tilde{g}_t \mathbb{1}_{t \leq \tau} + \tilde{g}_t(\tau) \mathbb{1}_{\tau < t}, \quad t \geq 0,
\]

where \(\tilde{g}\) is \(\mathbb{F}\)-predictable, positive or bounded, and \((t, \omega, u) \mapsto \tilde{g}_t(\omega, u)\) is a \(\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)\)-measurable positive or bounded function, to be identified.

- On the predictable set \(\{\tau < t\}\), being \(Y^\tau\) a \(G^\tau\)-predictable, positive or bounded, process (recall Proposition 1.2 (ii)), we immediately find \(\tilde{g}(\tau) = y(\tau)\).
• On the complementary set \( \{ t \leq \tau \} \), introducing the \( G \)-predictable process

\[
Y := (p^G Y^\tau)
\]

it is possible to use Remark 4.5, page 64 of Jeulin [17] (see also Dellacherie and Meyer [10], Ch. XX, page 186), to write

\[
Y_{1 \mid [0,\tau]} = \frac{1}{G^-} (p^F (Y_{1 \mid [0,\tau]}))_{1 \mid [0,\tau]} = \frac{1}{G^-} (p^G(Y^\tau))_{1 \mid [0,\tau]}.
\]

We then have, being \( 1_{[0,\tau]} \), by definition, \( G \)-predictable (recall that \( \tau \) is a \( G \)-stopping time),

\[
Y_{1 \mid [0,\tau]} = \frac{1}{G^-} (p^F(Y^\tau)_{1 \mid [0,\tau]}),
\]

where the last equality follows by the definition of predictable projection, being \( F \subset G \). Finally, given the result in Proposition 1.6 we have

\[
(p^F(Y^\tau)_{1 \mid [0,\tau]})_t = \int_t^{+\infty} y_t(u)p_{t-}(u)\nu(du)
\]

and the proposition is proved. \( \square \)

2 Martingales’ characterization

The aim of this section is to characterize \((\mathbb{P}, G^\tau)\) and \((\mathbb{P}, G)\)-martingales in terms of \((\mathbb{P}, F)\)-martingales. The analogous results under \( \mathbb{P}^* \) will follow as special cases.

**Proposition 2.1 Characterization of \((\mathbb{P}, G^\tau)\)-martingales in terms of \((\mathbb{P}, F)\)-martingales**

A process \( Y^\tau = y(\tau) \) is a \((\mathbb{P}, G^\tau)\)-martingale if and only if \( (y_t(u)p_t(u), t \geq 0) \) is a \((\mathbb{P}, F)\)-martingale, for \( \nu\)-almost every \( u \geq 0 \).

**Proof.** The sufficiency is a direct consequence of Proposition 1.2 and Lemma 1.4 (ii). Conversely, assume that \( y(\tau) \) is a \( G^\tau \)-martingale. Then, for \( s \leq t \), from Lemma 1.4 (ii),

\[
y_s(\tau) = \mathbb{E}(y_t(\tau)|G^\tau_s) = \frac{1}{p_s(\tau)} \mathbb{E}(y_t(u)p_t(u)|F_s)_{|u=\tau}
\]

and the result follows from Lemma 1.4 (i). \( \square \)

Passing to the progressive enlargement setting, we state and prove a martingale characterization result, essentially established by El Karoui et al. in [11] (see Theorem 5.7).

**Proposition 2.2 Characterization of \((\mathbb{P}, G)\) martingales in terms of \((\mathbb{P}, F)\)-martingales**

A \( G \)-adapted process \( Y_t := \hat{y}_t 1_{t<\tau} + \hat{y}_t(\tau)1_{\tau \leq t}, t \geq 0 \), is a \((\mathbb{P}, G)\)-martingale if and only if the following two conditions are satisfied

(i) for \( \nu\)-almost every \( u \geq 0 \), \( (\hat{y}_t(u)p_t(u), t \geq u) \) is a \((\mathbb{P}, F)\)-martingale;
(ii) the process \((\tilde{y}_t G_t + \int_0^t \tilde{y}_u(u)p_u(u)\nu(du), t \geq 0)\) is a \((\mathbb{P}, F)\)-martingale.

Proof. For the necessity, in a first step, we show that we can reduce our attention to the case where \(Y\) is u.i. (uniformly integrable); indeed, let \(Y\) be a \((\mathbb{P}, \mathbb{G})\)-martingale. For any \(T \geq 0\), let \(Y^{(T)} = (Y_t \wedge T, t \geq 0)\) be the associated stopped martingale, which is u.i. Assuming that the result is established for u.i. martingales will prove that the processes in (i) and (ii) are martingales up to time \(T\). Since \(T\) can be chosen as large as possible, we shall have the result.

Assume that \(Y\) is a u.i. \((\mathbb{P}, \mathbb{G})\)-martingale. From Proposition 1.1, \(Y_t = \mathbb{E}(Y_t^\tau | G_t)\) for some \((\mathbb{P}, \mathbb{G}^\tau)\)-martingale \(Y^\tau\). Proposition 2.1, then, implies that \(Y_t^\tau = y_t(\tau)\), where for \(\nu\)-almost every \(u \geq 0\) the process \((y_t(u)p_u(u), t \geq 0)\) is a \((\mathbb{P}, \mathbb{F})\)-martingale. Obviously, for \(t \geq u\), \(y_t(u) = \tilde{y}_t(u)\), and (i) is proved. We then have,

\[
\tilde{y}_t 1_{t < \tau} + \tilde{y}_t(\tau) 1_{t \leq \tau} = Y_t = \mathbb{E}(Y_t^\tau | G_t) = 1_{t < \tau} \frac{1}{G_t} \int_t^\infty y_t(u)p_u(u)\nu(du) + 1_{t \leq \tau} \tilde{y}_t(\tau)
\]

where the last equality results from the \((\mathbb{P}, \mathbb{F})\)-martingale property of the process \(y(u)p(u)\), for \(\nu\)-almost every \(u \geq 0\). We deduce

\[
\tilde{y}_t G_t = \mathbb{E} \left( \int_t^\infty y_u(u)p_u(u)\nu(du) | F_t \right) = \mathbb{E} \left( \int_t^\infty y_u(u)p_u(u)\nu(du) | F_t \right) - \int_0^t y_u(u)p_u(u)\nu(du),
\]

which implies (in view of \(y_t(u) = \tilde{y}_t(u)\), for \(t \geq u\)) that \((\tilde{y}_t G_t + \int_0^t \tilde{y}_u(u)p_u(u)\nu(du), 0 \leq t \leq T)\) is a \((\mathbb{P}, \mathbb{F})\)-martingale and (ii) immediately follows.

Conversely, assuming (i) and (ii), we verify \(\mathbb{E}(Y_t | G_s) = Y_s\) for \(s \leq t\). Indeed,

\[
\mathbb{E}(Y_t | G_s) = \mathbb{E}(1_{s < \tau} \tilde{y}_t + 1_{s < \tau \leq \tau} \tilde{y}_t(\tau) | G_s) + \mathbb{E}(1_{\tau \leq \tau} \tilde{y}_t(\tau) | G_s) = 1_{s < \tau} \frac{1}{G_s} \mathbb{E}(\tilde{y}_t u(u)p_u(u) | F_s) + 1_{\tau \leq \tau} \frac{1}{p_\tau(u)} \mathbb{E}(\tilde{y}_t(u)p_u(u) | F_s) | u = \tau,
\]

where we have used Lemma 3.1.2 in Bielecki et al. [6], and Lemma 1.5 to obtain the last equality.

Next, using condition (i), it follows that

\[
\mathbb{E}(Y_t | G_s) = 1_{s < \tau} \frac{1}{G_s} \mathbb{E}(\tilde{y}_t G_t + \int_s^t \tilde{y}_u(u)p_u(u)\nu(du) | F_s) + 1_{\tau \leq \tau} \frac{1}{p_\tau(u)} \tilde{y}_s(u)p_s(u)
\]

\[
= 1_{s < \tau} \frac{1}{G_s} \mathbb{E}(\tilde{y}_t G_t + \int_s^t \tilde{y}_u(u)p_u(u)\nu(du) | F_s) - 1_{s < \tau} \frac{1}{G_s} \int_0^s \tilde{y}_u(u)p_u(u)\nu(du) + 1_{\tau \leq \tau} \tilde{y}_s(\tau)
\]

\[
= 1_{s < \tau} \frac{1}{G_s} \tilde{y}_s G_s + 1_{\tau \leq \tau} \tilde{y}_s(\tau) = Y_s,
\]

where we used condition (ii) to obtain the next-to-last identity.

We end this section with a curiosity linking martingales in the filtrations \(\mathbb{G}\) and \(\mathbb{G}^\tau\); we already know, from Remarks 1.2, that any \((\mathbb{P}^*, \mathbb{F})\)-martingale remains a \((\mathbb{P}^*, \mathbb{G}^\tau)\)-martingale, but it is not true that any \((\mathbb{P}^*, \mathbb{G})\)-martingale remains a \((\mathbb{P}^*, \mathbb{G}^\tau)\)-martingale. Indeed, we have the following result.

Lemma 2.3 Any \((\mathbb{P}^*, \mathbb{G})\)-martingale \(X^*\) is a \((\mathbb{P}^*, \mathbb{G}^\tau)\) semi-martingale which can have a non-null bounded variation part.
Proof. In all the proof, we work under $\mathbb{P}^*$. The result follows immediately from Proposition 2.2 (under $\mathbb{P}^*$), noticing that the $(\mathbb{P}^*, \mathbb{G})$ martingale $Y^*$ can be written as $Y^*_t = \tilde{y}^*_t 1_{t<\tau} + \tilde{y}^*_t(\tau) 1_{t\leq \tau}$.

Therefore, in the filtration $\mathbb{G}^\tau$, it is the sum of two $\mathbb{G}^\tau$ semi-martingales: the processes $1_{t<\tau}$ and $1_{t\leq \tau}$

are $\mathbb{G}^\tau$ semi-martingales, as well as the processes $\tilde{y}, \tilde{y}^*(\tau)$. Indeed, from Proposition 2.2, recalling that the $(\mathbb{P}^*, \mathbb{F})$-density of $\tau$ is a constant equal to one, we know that, for every $u > 0$, $(\tilde{y}^*_t(u), t \geq u)$

is an $\mathbb{F}$-martingale and that the process $(\tilde{y}^*_t G(t) + \int_0^t \tilde{y}^*_u(u) \nu(du), t \geq 0)$ is an $\mathbb{F}$-martingale, hence $\tilde{y}^*$ is a $\mathbb{G}$-semi-martingale).

It can be noticed that the $(\mathbb{P}^*, \mathbb{G})$-martingale $M^*$, is such that $M^*_t$ is, for any $t$, a $\mathbb{G}_0^\tau$-measurable random variable. Therefore, $M^*$ is not a $(\mathbb{P}^*, \mathbb{G}^\tau)$-martingale, since, for $s \leq t$, $\mathbb{E}(M^*_s | \mathbb{G}_0^\tau) = M^*_t \neq M^*_s$, but it is a bounded variation $\mathbb{G}^\tau$-predictable process, hence a $\mathbb{G}^\tau$-semi-martingale with null martingale part. In other terms, $\mathbb{H}$ is not immersed in $\mathbb{G}^\tau$ under $\mathbb{P}^*$. \hfill \Box

3 Canonical decomposition

In this section, we work under $\mathbb{P}$ and we show that any $\mathbb{F}$-local martingale $x$ is a semi-martingale in both the initially enlarged filtration $\mathbb{G}^\tau$ and in the progressively enlarged filtration $\mathbb{G}$, and that any $\mathbb{G}$-martingale is a $\mathbb{G}^\tau$-semi-martingale. We also provide the canonical decomposition of any $\mathbb{F}$-local martingale as a semi-martingale in $\mathbb{G}^\tau$ and in $\mathbb{G}$. Under the assumption that the $\mathbb{F}$-conditional law of $\tau$ is absolutely continuous w.r.t. the law of $\tau$, these questions were answered in Jacod [15], in the initial enlargement setting, and in Jeamblanc and Le Cam [16] and El Karoui et al. [11], in the progressive enlargement case. Our aim here is to retrieve their results in an alternative manner.

We will need the following technical result, concerning the existence of the predictable bracket $\langle x, p \rangle (u)$. From Theorem 2.5 a) in Jacod [15], it follows immediately that, under the $(\mathcal{E})$-Hypothesis, for every $(\mathbb{P}, \mathbb{F})$-(local)martingale $x$, there exists a $\nu$-negligible set $B$ (depending on $x$), such that $\langle x, p \rangle (u)$ is well-defined for $u \notin B$. Hereafter, by $\langle x, p \rangle (u)$, we mean $\langle x, p \rangle (u) |_{u \in \tau}$.

Furthermore, according to Theorem 2.5 b) in Jacod [15], under the $(\mathcal{E})$-Hypothesis, there exists an $\mathbb{F}$-predictable increasing process $A$ and a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$-measurable function $(t, \omega, u) \mapsto k_t(\omega, u)$ such that, for any $u \notin B$ and for all $t \geq 0$,

$$\langle x, p \rangle (u)_t = \int_0^t k_s(u) p_{s-}(u) dA_s \ \text{a.s.}$$

(7)

(the two processes $A$ and $k$ depend on $x$, however, to keep simple notation, we do not write $A(x)$ nor $k(x)$).

Moreover,

$$\int_0^t |k_s(\tau)| dA_s < \infty \ \text{a.s., for any } t > 0.$$  \hfill (8)

The following two propositions provide, under the $(\mathcal{E})$-Hypothesis, respectively, the canonical decomposition of any $(\mathbb{P}, \mathbb{F})$-local martingale $x$ in the enlarged filtrations $\mathbb{G}^\tau$ and $\mathbb{G}$. The first result is due to Jacod [15, Theorem 2.5 c)]. Our proof is different, but less general.

Proposition 3.1 Canonical Decomposition in $\mathbb{G}^\tau$

Any $(\mathbb{P}, \mathbb{F})$-local martingale $x$ is a $(\mathbb{P}, \mathbb{G}^\tau)$-semimartingale with canonical decomposition

$$x_t = X^*_t + \int_0^t \frac{d\langle x, p \rangle (u)}{p_{s-}(\tau)}.$$
for some \((P, G^\tau)\)-local martingale \(X^\tau\).

**Proof.** In view of Remark 1.2, if \(x\) is a \((P,F)\)-martingale, it is a \((P^*, G^\tau)\)-martingale, too. Noting that \(\frac{dF}{dP} = p_t(\tau)\) on \(G^\tau_t\), Girsanov’s theorem tells us that the process \(X^\tau\), defined by

\[
X^\tau_t = x_t - \int_0^t \frac{d(x,p(\tau))_s}{p_s(\tau)}
\]

is a \((P, G^\tau)\)-martingale. \(\Box\)

Now, any \(F\)-local martingale is a \(G\)-adapted process and a \(G^\tau\) semi-martingale (from the above Proposition 3.1), so in view of Stricker’s Theorem [22], it is also a \(G\) semi-martingale. The following proposition aims to obtain the \(G\)-canonical decomposition of an \(F\)-local martingale. We refer to Jeanblanc and Le Cam [16] for an alternative proof.

We need some preliminary results.

The Azéma super-martingale \(G\), introduced in Equation (3), admits the Doob-Meyer decomposition \(G_t = \mu_t - \int_0^t p_u(u)\nu(du), t \geq 0\), where \(\mu\) is the \(F\)-martingale defined as

\[
\mu_t := 1 - \int_0^t (p_t(u) - p_u(u))\nu(du)
\]

(see, e.g., Section 4.2.1 in El Karoui et al. [11]). The following lemma provides a formula for the predictable quadratic covariation process \(\langle x, G \rangle = \langle x, \mu \rangle\) in terms of the density \(p\).

**Lemma 3.2** Let \(x\) be an \(F\)-martingale and \(\mu\) the \(F\)-martingale part in the Doob-Meyer decomposition of \(G\). If \(kp^-\) is \(dA \otimes du\)-integrable, then

\[
\langle x, \mu \rangle_t = \int_0^t dA_s \int_s^\infty \nu(du)k_s(u)p_{s-}(u),
\]

where \(k\) was introduced in Equation (7).

**Proof.** First consider the right-hand-side of (9), that is by definition predictable, and apply Fubini’s Theorem

\[
\xi_t := \int_0^t dA_s \int_s^\infty k_s(u)p_{s-}(u)\nu(du)
\]

\[
= \int_0^t dA_s \int_0^t k_s(u)p_{s-}(u)\nu(du) + \int_0^t dA_s \int_t^\infty k_s(u)p_{s-}(u)\nu(du)
\]

\[
= \int_0^t \nu(du) \int_0^u k_s(u)p_{s-}(u)dA_s + \int_0^t \nu(du) \int_t^\infty k_s(u)p_{s-}(u)dA_s
\]

\[
= \int_0^t \langle x, p(u) \rangle_u \nu(du) + \int_t^\infty \langle x, p(u) \rangle_t \nu(du)
\]

\[
= \int_0^\infty \langle x, p(u) \rangle_t \nu(du) + \int_0^t (\langle x, p(u) \rangle_u - \langle x, p(u) \rangle_t) \nu(du).
\]

To verify (9), it suffices to show that the process \(x\mu - \xi\) is an \(F\)-local martingale (since \(\xi\) is a predictable, finite variation process). By definition, for \(\nu\)-almost every \(u \in \mathbb{R}^+\), the process
\[(m_t)(u) := x_t p_t(u) - \langle x, p(u) \rangle_t, t \geq 0 \] is an \( \mathbb{F} \)-local martingale. Then, given that \( 1 = \int_0^\infty p_t(u) \nu(du) \) for every \( t \geq 0 \), a.s., we have

\[
x_t \xi_t = x_t \int_0^\infty p_t(u) \nu(du) - x_t \int_0^t (p_t(u) - p_u(u)) \nu(du)
- \int_0^\infty \langle x, p(u) \rangle_t \nu(du) + \int_0^t (\langle x, p(u) \rangle_t - \langle x, p(u) \rangle_u) \nu(du)
= \int_0^\infty m_t(u) \nu(du) - \int_0^t (m_t(u) - m_u(u)) \nu(du) + \int_0^t p_u(u)(x_t - x_u) \nu(du) .
\]

The first two terms are martingales (because of the martingale property of \( m(u) \)). As for the last term, using the fact that \( \nu \) has no atoms, we find

\[
d \left( x_t \int_0^t p_u(u) \nu(du) - \int_0^t p_u(u)x_u \nu(du) \right) = \int_0^t p_u(u) \nu(du) dx_t + x_t p_t(t) \nu(dt) - p_t(t)x_t \nu(dt)
= \int_0^t p_u(u) \nu(du) dx_t
\]
and we have, indeed, proved that \( x \mu - \xi \) is an \( \mathbb{F} \)-local martingale.

**Proposition 3.3 Canonical Decomposition in \( \mathcal{G} \)**

Any (càdlàg) \((\mathbb{P}, \mathbb{F})\)-local martingale \( x \) is a \((\mathbb{P}, \mathcal{G})\) semi-martingale with canonical decomposition

\[
x_t = X_t + \int_0^{t \wedge \tau} \frac{d\langle x, G_s \rangle_s}{G_{s-}} + \int_0^{t \wedge \tau} \frac{d\langle x, p(\tau) \rangle_s}{p_{s-}(\tau)}
\]

where \( X \) is a \((\mathbb{P}, \mathcal{G})\)-local martingale.

**Proof.** First of all let us recall that if the reference filtration \( \mathbb{F} \) is right-continuous, any \( \mathbb{F} \)-martingale has a càdlàg version. We consider, then, here, this càdlàg version of the martingale \( x \).

From Proposition 3.1, any \( \mathbb{F} \)-local martingale \( x \) can be decomposed as \( x = X^\tau + C \) where \( X^\tau \) is a \((\mathbb{P}, \mathbb{G}^\tau)\)-local martingale and (recall Equation (7))

\[
C_t = \int_0^t \frac{d\langle x, p(\tau) \rangle_s}{p_{s-}(\tau)} = \int_0^t k_s(\tau) dA_s .
\]

The idea is to project this decomposition on the filtration \( \mathcal{G} \):

\[
x_t = \langle x \rangle_t = \langle x \rangle_t^{(o)} X_t + \langle x \rangle_t^{(o)} C_t = C_t^{(p)} + C_t^{(p)} ,
\]

where the left-hand side superscript \( "(o)" \) indicates the optional projection (on \( \mathcal{G} \)) and where the right-hand side superscript \( "(p)" \) denotes the dual predictable projection (on \( \mathcal{G} \)). Now, the process \( X := \langle x \rangle_t + (\langle x \rangle_t - C_t^{(p)}) \) is the sum of two \( \mathcal{G} \)-martingales, hence it is a \( \mathcal{G} \)-martingale. From classical results on the predictable projection of processes (see for instance Theorem 57, Chapter VI of [9]), we also have, being \( A \) predictable,

\[
C_t^{(p)} = \int_0^t v(s) k(\tau) dA_s .
\]
From Proposition 1.7, moreover, $k(\tau)$ being $G^\tau$-predictable

$$
(p) (k(\tau))_s = 1_{s \leq \tau} \frac{1}{G^-} \int_s^\infty k_s(u)p_{s-}(u)\nu(du) + 1_{\tau < s}k_s(\tau) .
$$

Thus, substituting (12) in (11) and using Lemma 3.2, one obtains decomposition (10).

As in Lemma 2.3, we deduce that any $(P, G)$-martingale is a $(P, G^\tau)$ semi-martingale.

Note that it is also possible to prove this result from Lemma 2.3 and a change of probability.

## 4 Predictable Representation Theorems

The aim of this section is to obtain Predictable Representation Theorem (PRT hereafter) in the enlarged filtrations $G$ and $G^\tau$, both under $P$ and $P^\ast$. We start by assuming that there exists a $(P, F)$-local martingale $z$ (eventually multidimensional), such that the Predictable Representation Property (PRP hereafter) holds in $(P, F)$. Notice that $z$ is not necessarily continuous.

Beforehand we introduce some notation: $M_{\text{loc}}(P, F)$ denotes the set of $(P, F)$-local martingales, while $M^2(P, F)$ denotes the set of $(P, F)$-martingales $x$, such that

$$
\mathbb{E}(x_t^2) < \infty, \ \forall \ t \geq 0.
$$

Also, for a $(P, F)$-local martingale $m$, we denote by $L(m, P, F)$ the set of $F$-predictable processes which are integrable with respect to $m$ (in the sense of local martingale), namely (see, e.g., Definition 9.1 and Theorem 9.2. in [14])

$$
L(m, P, F) = \left\{ \varphi \in \mathcal{P}(F) : \left( \int_0^t \varphi_s^2 \sigma[m]_s \right)^{1/2} \text{ is } P - \text{locally integrable} \right\}.
$$

### Assumption 4.1 PRT for $(P, F)$

There exists a process $z \in M_{\text{loc}}(P, F)$ such that every $x \in M_{\text{loc}}(P, F)$ can be represented as

$$
x_t = x_0 + \int_0^t \varphi_s dz_s
$$

for some $\varphi \in L(z, P, F)$.

We start investigating what happens under the measure $P^\ast$, in the initially enlarged filtration $G^\tau$.

Notice that, under the equivalence assumption in $[0, T]$ and assuming a PRT in the reference filtration $F$, Amendinger (see [1, Th. 2.4]) proved a PRT in $(P^\ast, G^\tau)$. This result was extended to $(P, G^\tau)$, in the case where the underlying (local) martingale in the reference filtration is continuous.

### Proposition 4.1 PRT for $(P^\ast, G^\tau)$

Under Assumption 4.1, every $X^\tau \in M_{\text{loc}}(P^\ast, G^\tau)$ admits a representation

$$
X^\tau_t = X^\tau_0 + \int_0^t \Phi^\tau_s dz_s
$$

where $\Phi^\tau \in L(z, P^\ast, G^\tau)$.

In the case where $X^\tau \in M^2(P^\ast, G^\tau)$ and $z \in M^2(P, F)$, one has

$$
\mathbb{E}^\ast \left( \int_0^t (\Phi^\tau_s)^2 d|z|_s \right) < \infty, \text{ for all } t \geq 0 \text{ and the representation is unique}.
$$
Proof.

From Theorem 13.4 in [14], it suffices to prove that any bounded martingale admits a predictable representation in terms of \( z \). Let \( X^\tau \in \mathcal{M}^{\text{loc}}(\mathbb{P}^*, \mathbb{G}^\tau) \) be bounded by \( K \). From Proposition 2.1, \( X^\tau = x_t(\tau) \) where for \( \nu \)-almost every \( u \in \mathbb{R}^+ \), the process \( \{x_t(u), t \geq 0\} \) is a \((\mathbb{P}^*, \mathcal{F})\)-martingale, hence a \((\mathbb{P}, \mathcal{F})\)-martingale. Thus (for \( \nu \)-almost every \( u \in \mathbb{R}^+ \)), Assumption 4.1 implies that

\[
x_t(u) = x_0(u) + \int_0^t \varphi_s(u)dz_s,
\]

where \( \varphi_t(u), t \geq 0 \) is an \( \mathcal{F} \)-predictable process.

The process \( X^\tau \) being bounded by \( K \), it follows by an application of Lemma 1.4(i) that for \( \nu \)-almost every \( u \geq 0 \), the process \( \{x_t(u), t \geq 0\} \) is bounded by \( K \). Then, using the Itô isometry,

\[
\mathbb{E}^\tau \int_0^t \varphi_s^2(u)dz_s = \mathbb{E}^\tau \left( \int_0^t \varphi_s(u)dz_s \right)^2
\]

\[
= \mathbb{E}^\tau ((x_t(u) - x_0(u))^2) = \mathbb{E}^\tau (x_t^2(u)) \leq K^2.
\]

Also, from [23, Lemma 2], one can consider a version of the process \( \int_0^t \varphi_s^2(u)dz_s \), which is measurable with respect to \( u \). Using this fact,

\[
\mathbb{E}^\tau \left( \left( \int_0^t \varphi_s^2(u)dz_s \right)^{1/2} \right)^2 = \mathbb{E}^\tau \left( \int_0^t \varphi_s^2(u)dz_s \right) \leq \mathbb{E}^\tau (x_t^2(u)) \leq K^2.
\]

The process \( \Phi^\tau \) defined by \( \Phi^\tau_t = \varphi_t(\tau) \) is \( \mathbb{G}^\tau \)-predictable, according to Proposition 1.2, it satisfies (14), with \( X_0 = x_0(\tau) \), and it belongs to \( \mathcal{L}(z, \mathbb{P}^*, \mathbb{G}^\tau) \).

If \( X^\tau \in \mathcal{M}^2(\mathbb{P}^*, \mathbb{G}) \) and if \( z \in \mathcal{M}^2(\mathbb{P}, \mathcal{F}) \) (or equivalently, \( z \in \mathcal{M}^2(\mathbb{P}^*, \mathcal{F}) \)), from Itô’s isometry,

\[
\mathbb{E}^\tau \int_0^t (\Phi^\tau_s)^2dz_s = \mathbb{E}^\tau \left( \int_0^t \Phi^\tau_s dz_s \right)^2 = \mathbb{E}^\tau (X_t^\tau - X_0^\tau)^2 < \infty.
\]

Also, from this last equation, if \( X^\tau \equiv 0 \) then \( \Phi^\tau \equiv 0 \) from which the uniqueness of representation follows. \hfill \Box

Passing to the progressively enlarged filtration \( \mathbb{G} \), which consists of two filtrations, \( \mathbb{G} = \mathcal{F} \vee \mathcal{H} \), intuitively one needs two martingales to establish a PRT. Apart from \( z \), intuition tells us that a candidate for the second martingale might be the compensated martingale of \( H \), that was introduced, respectively under \( \mathbb{F} \) (it was denoted by \( M \)) and under \( \mathbb{P}^* \) (denoted by \( M^* \)), in Equation (5) and in Equation (6).

**Proposition 4.2 PRT for \((\mathbb{P}^*, \mathbb{G})\)**

Under Assumption 4.1, every \( X \in \mathcal{M}^{\text{loc}}(\mathbb{P}^*, \mathbb{G}) \) admits a representation

\[
X_t = X_0 + \int_0^t \Phi_s dz_s + \int_0^t \Psi_s dM^*_s
\]

for some processes \( \Phi \in \mathcal{L}(z, \mathbb{P}^*, \mathbb{G}) \) and \( \Psi \in \mathcal{L}(M^*, \mathbb{P}^*, \mathbb{G}) \). Moreover, if \( X \in \mathcal{M}^2(\mathbb{P}^*, \mathbb{G}) \) and \( z \in \mathcal{M}^2(\mathbb{P}^*, \mathcal{F}) \), one has, for any \( t \geq 0 \),

\[
\mathbb{E}^\tau \left( \int_0^t \Phi^2_s ds \right) < \infty, \quad \mathbb{E}^\tau \left( \int_0^t \Psi^2_s \lambda^*(s) \nu(ds) \right) < \infty,
\]

and the representation is unique.
Proof. It is known that any \((P^*, H)\) local martingale \(\xi\) can be represented as \(\xi_t = \xi_0 + \int_0^t \psi_s dM^*_s\) for some process \(\psi \in \mathcal{L}(M^*, P^*, H)\) (see, e.g., the proof in Chou and Meyer [8]). Notice that \(\psi\) has a role only before \(\tau\) and, for this reason (recall that \(H = (H_t)_{t \geq 0}\), where \(H_t = \sigma(\tau \wedge t)\)), \(\psi\) can be chosen deterministic.

Under \(P^*\), we then have

- the PRP holds in \(\mathbb{F}\) with respect to \(z\),
- the PRP holds in \(\mathbb{H}\) with respect to \(M^*\),
- the filtration \(\mathbb{F}\) and \(\mathbb{H}\) are independent.

From classical literature (see Lemma 9.5.4.1(ii) of Jeanblanc, Yor and Chesney [7], for instance) the filtration \(G = \mathbb{F} \vee H\) enjoys the PRP under \(P^*\) with respect to the pair \((z, M^*)\).

Now suppose that \(X \in M^2(P^*, G)\). We find

\[
\infty > \mathbb{E}^*(X_t - X_0)^2 = \mathbb{E}^* \left( \int_0^t \Phi_s dZ_s + \int_0^t \Psi_s dM^*_s \right)^2 \\
= \mathbb{E}^* \int_0^t \Phi_s^2 d[Z]_s + 2 \mathbb{E}^* \left( \int_0^t \Phi_s dZ_s \int_0^t \Psi_s dM^*_s \right) + \mathbb{E}^* \int_0^t \Psi_s \lambda^*(s) \nu(ds),
\]

where in the last equality we used the Itô isometry. The cross-product term in the last equality is zero due to the orthogonality of \(z\) and \(M^*\) (under \(P^*\)). From this inequality, the desired integrability conditions hold and the uniqueness of the representation follows (as in the previous proposition). \(\square\)

Remark 4.1 In order to establish a PRT for the initially enlarged filtration \(G^\tau\) and under \(P^*\), one could have proceeded as in the proof of Proposition 4.2, noting that any martingale \(\xi\) in the “constant" filtration \(\sigma(\tau)\) satisfies \(\xi_t = \xi_0 + 0\) and that under \(P^*\) the two filtrations \(\mathbb{F}\) and \(\sigma(\tau)\) are independent.

Proposition 4.3 PRT under \(P\)

Under Assumption 4.1, one has:

(i) Every \(X^\tau \in M^{loc}(P, G^\tau)\) can be represented as

\[
X^\tau_t = X^\tau_0 + \int_0^t \Phi^\tau_s dZ^\tau_s
\]

where \(Z^\tau\) is the martingale part in the \(G^\tau\)-canonical decomposition of \(z\) and \(\Phi \in \mathcal{L}(Z^\tau, P, G^\tau)\).

(ii) Every \(X \in M^{loc}(P, G)\) can be represented as

\[
X_t = X_0 + \int_0^t \Phi_s dZ_s + \int_0^t \Psi_s dM_s,
\]

where \(Z\) is the martingale part in the \(G\)-canonical decomposition of \(z\), \(M\) is the \((P, G)\)-compensated martingale associated with \(H\) and \(\Phi \in \mathcal{L}(Z, P, G)\) and \(\Psi \in \mathcal{L}(M, P, G)\).

Proof. The assertion (i) (resp. (ii)) follows from Proposition 4.1 (resp. Proposition 4.2) and the stability of PRP under an equivalent change of measure (see for example Theorem 13.12 in [14]).
For part (ii), it is important to note that, if \( z \) is a \((\mathbb{P}, F)\)-martingale, it is a \((\mathbb{P}^*, G)\)-martingale, too. Hence, by a Girsanov type transformation, \( Z \) defined as \( dZ_t := dz_t - \frac{1}{\ell^*}d(z, \ell^*)_t \), \( Z_0 = z_0 \), is a \((\mathbb{P}, G)\)-martingale, where \( \ell^* := 1/\ell \) is a \((\mathbb{P}^*, G)\)-martingale (in fact \( d\mathbb{P}_{\mid \mathcal{G}_t} = \ell^*_t \, d\mathbb{P}^*_{\mid \mathcal{G}_t} \)). From the uniqueness of the canonical decomposition of the \((\mathbb{P}, G)\)-semimartingale \( z \) (which is indeed special) and from Proposition 3.3, it follows that the \((\mathbb{P}, G)\)-martingale \( Z \) is in particular given by

\[
Z_t = z_t - \int_0^{t \wedge \tau} \frac{d(z, G)_s}{G_{s-}} - \int_0^{t \wedge \tau} \frac{d(z, P(\tau))_s}{p_s(\tau)}. 
\]

\[\square\]

5 Concluding Remarks

- In the multi-dimensional case, that is when \( \tau = (\tau_1, \cdots, \tau_d) \) is a vector of finite random times, the same machinery can be applied. More precisely, under the assumption

\[
\mathbb{P}(\tau_1 \in \theta_1, \cdots, \tau_d \in \theta_d \mid \mathcal{F}_t) \sim \mathbb{P}(\tau_1 \in \theta_1, \cdots, \tau_d \in \theta_d)
\]

one defines the probability \( \mathbb{P}^* \) on \( \mathcal{G}_t = \mathcal{F}_t \vee \sigma(\tau_1) \vee \cdots \vee \sigma(\tau_d) \) with respect to \( \mathbb{P} \) by

\[
\frac{d\mathbb{P}^*}{d\mathbb{P} \mid \mathcal{G}_t} = \frac{1}{p_t(\tau_1, \cdots, \tau_d)},
\]

where \( p_t(\tau_1, \cdots, \tau_d) \) is the (multidimensional) analog to \( p_t(\tau) \), and the results for the initially enlarged filtration are obtained in the same way as for the one-dimensional case.

As for the progressively enlarged filtration, one has to note that, in this case, a measurable process is decomposed into \( 2^d \) terms, depending on whether \( t < \tau_i \) or \( \tau_i \leq t \).

- Notice that honest times (recall that a random time \( L \) is honest if it is equal to an \( \mathcal{F}_t \)-measurable random variable on \( L < t \)) cannot be included in this study. Indeed, it was shown by Nikeghbali and Yor in [18], Theorem 4.1, that, in the case where all \( \mathbb{F} \)-martingales are continuous and if the honest time \( L \) avoids \( \mathbb{F} \)-stopping times, then there exists a continuous and nonnegative local martingale \( (N_t)_{t \geq 0} \), with \( N_0 = 1 \) and \( \lim_{t \to +\infty} N_t = 0 \), such that:

\[
\mathbb{P}(L > t \mid \mathcal{F}_t) = \frac{N_t}{S_t},
\]

where \( S_t := \sup_{s \leq t} N_s \). In our case, under the \((\mathcal{E})\)-Hypothesis, the above equation does not hold true, since the Azéma supermartingale \( G \) admits the multiplicative decomposition \( G_t = N_t e^{-\Lambda_t} \), where the intensity process \( \Lambda \) is strictly increasing (so that it is not possible that \( e^{\Lambda_t} = \sup_{s \leq t} N_s \)).

Another reason to say that, if random time \( \tau \) do not belong to \( \mathcal{F}_\infty \), it is not an honest time. Assuming now that \( \tau \) belongs to \( \mathcal{F}_\infty \), under the probability \( \mathbb{P}^* \), it is independent of \( \mathbb{F} \), thus \( \mathbb{F} \) is immersed in \( G \) (see Proposition ??). So, in particular, \( \tau \) is a pseudo-stopping time, that is for every \( \mathbb{F} \)-martingale \( x \), the process \( (x_{t \wedge \tau}, t \geq 0) \) is a \( G \)-martingale. But, according to Nikeghbali and Yor [19, Proposition 6], an honest time is a pseudo-stopping time only if it is a stopping time (note that under \((\mathcal{E})\)-Hypothesis, \( \tau \) avoids \( \mathbb{F} \)-stopping times).

- Under immersion property and under the \((\mathcal{E})\)-Hypothesis, \( p_t(u) = p_s(u), t \geq u \). In particular, as expected, the canonical decomposition’s formulae presented in Section 3 are trivial.
• Predictable representation theorems can be obtained in the more general case, where any \((P,F)\)-martingale \(x\) admits a representation as

\[
x_t = x_0 + \int_0^t \int_E \varphi(s, \theta) \tilde{\mu}(ds, d\theta),
\]

for a compensated martingale associated with a point process.

References


