Optimal Investment Problems with Uncertain Time Horizon

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October 6, 2010

Abstract

In this paper we consider an agent on a financial market who can trade with an uncertain time horizon by investing in risky stocks and a risk-free bond. He aims at maximizing the utility he draws from his final wealth measured by some utility function. We obtain a sufficient and necessary condition for the optimality, which gives an explicit expression for the optimal strategies as solutions of a new type of forward-backward stochastic differential equation (FBSDE). We also give an existence and uniqueness result for this kind of FBSDEs.

Key words: Uncertain time horizon; Forward-backward stochastic differential equation; Maximum principle

AMS subject classification: 60H10, 91B28, 93E20.

1 Introduction

Most of standard financial economics is based on the assumption that, at the moment of making an investment decision, an investor knows with certainty the time of eventual exit. Such an assumption can be traced back to the origins of modern financial economics, and, in particular, to the development of portfolio selection theory by Markowitz \cite{9}. But in practice, most investors would acknowledge the fact that, upon entering the market, they never know with certainty the time of exiting the market. It is both of practical and theoretical interest to develop a comprehensive theory of optimal investment and consumption under uncertain time horizon. In most of paper on this subject, the random time horizon was assumed to be independent of all other sources of uncertainty, for example, Yaari \cite{13}, Hakansson \cite{3,4}, Morton \cite{10} and Richard \cite{12}. On the other hand, in the paper of Karatzas and Wang \cite{7}, the randomness of time horizon was fully dependent upon asset price and induced no new uncertainty in the economy. This was the other extremely stylized assumption. Blanchet-Scalliet, El Karoui, Jeanblanc and Martellini \cite{1} first considered a general situation of the uncertainty of time horizon, which cover the above two extreme assumptions. In their paper, an sufficient condition for an optimal investment problem in the presence of an uncertain time horizon was obtained by martingale approach, and was applied to solve the optimal investment problem in a setup with constant relative risk aversion (CRRA) preferences, constant expected return and drift parameters, and a deterministic distribution function of time horizon. This problem with CRRA preferences was also studied through backward stochastic differential equation (BSDE) approach in El Karoui, Jeanblanc and Huang \cite{2} under some suitable assumptions.

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In this paper, we study the optimal investment problems with a general uncertain time horizon formulated by [1] [2]. Applying the classical stochastic maximum principle, we get a necessary condition for the optimality. With the help of the concavity property of utility functions, we show the necessary condition is sufficient also. This condition gives an explicit expression for the optimal strategies and the corresponding wealth processes as solutions of a new type of forward-backward stochastic differential equations (FBSDEs). Using an invertible transformation, we give an existence and uniqueness result for this kind of FBSDEs, which is out of the scope of the known results in the FBSDEs theory. We note that, comparing with Blanchet-Scalliet et al. [1], the method and results in this paper are effective without any deterministic assumptions.

The rest of the paper is organized as follows. In Section 2 we introduce the model of an economy with an uncertain time horizon. In Section 3 we give a sufficient and necessary condition for the problem of optimal dynamic investment decision in the presence of an uncertain time horizon. Section 4 is devoted to obtain an existence and uniqueness result of a new kind of FBSDEs.

2 Basic notations and problem formulation

In this section, we introduce a general model for the economy in the presence of an uncertain time horizon. Let $[0, T]$, with $T > 0$, denote the finite time span of the economy. Uncertainty in the economy is described through a probability space $(\Omega, \mathcal{A}, P)$ on which is defined an $n$-dimensional Brownian motion $W$.

2.1 Asset prices

The basic securities consist of $n + 1$ assets. One of them is a risk-free asset (the money market instrument or a default free bond) with price per unit $B$ governed by the equation

$$\frac{dB_t}{B_t} = r_t dt, \quad (1)$$

where $r$ is the interest rate. In addition to the bond, $n$ risky securities (the stock) are continuously traded. The price process $S^i$ for one share of the $i$-th stock is modeled by the linear stochastic differential equation

$$\frac{dS^i_t}{S^i_t} = \mu^i_t dt + \sum_{j=1}^{n} \sigma^i_j dW^j_t, \quad i = 1, 2, \ldots, n. \quad (2)$$

Agents’ basic information set is captured by the filtration $\mathbb{F} = \{\sigma(S_s, s \leq t); t \geq 0\}$, with $\mathcal{F}_T \subset \mathcal{A}$ and $\mathcal{F}_0$ is trivial. We make the following standard assumptions.

- the coefficients $\mu = (\mu^i)_{i=1,\ldots,n}$, $r$ are predictable and bounded, and $r_t > 0$;
- The coefficient $\sigma = (\sigma^{ij})_{i,j=1,\ldots,n}$ is predictable, bounded, invertible and the inverse $\sigma^{-1}$ is also bounded;
- $W = (W^i)_{i=1,\ldots,n}$ is an $\mathbb{F}$-Brownian motion.

Under these assumptions, the market is complete and arbitrage-free (see for example, Karatzas [6]). We denote by $\theta_t = \sigma_t^{-1}(\mu_t - r_t 1)$, $t \in [0, T]$, which is then also a predictable bounded process, where $1$ is the vector whose every component is 1.

Consider now an investor who invests at time $t$ the amount $\tilde{\pi}_t^i$ of the wealth $X_t$ in the $i$-th stock, $i = 1, \ldots, n$. If the strategy $\tilde{\pi} = (\tilde{\pi}^1, \tilde{\pi}^2, \ldots, \tilde{\pi}^n)^\top$, where the superscript $\top$ denotes the
transpose of vectors or matrices, is used in a self-financing way, i.e., if the wealth invested in
the riskless asset is $X_t - \sum_{i=1}^{n} \tilde{\pi}_i t$, then the wealth process $(X_t, t \geq 0)$, with $X_0 = x$, evolves
according to the following stochastic differential equation
\[ dX_t^\pi = [r_t X_t^\pi + \pi_t^\top \theta] dt + \pi_t^\top dW_t, \] (3)
where $\pi_t = \sigma_t^\top \tilde{\pi}_t$. Of course, the investor’s decisions can only be based on the current information
$F_t$, i.e., the processes $\pi$ (equivalently, $\tilde{\pi}$) is $F$-predictable. To study our problem, we give more
assumptions on $\pi$.

**Definition 2.1.** A portfolio $(\pi_t, 0 \leq t \leq T)$ is said to be admissible if it is $F$-predictable and
square integrable, i.e., such that
\[ \mathbb{E} \int_0^T |\pi_t|^2 dt < \infty. \]

Let $L^2_F(0, T; \mathbb{R}^m)$ denote the set of $F$-predictable $\mathbb{R}^m$-valued square integrable processes. Then
the set of admissible portfolio is $L^2_F(0, T; \mathbb{R}^n)$.

### 2.2 Timing uncertainty

In this paper, we assume that an agent’s time horizon $\tau$ is a positive random variable measurable
with respect to the sigma-algebra $A$. Importantly, we do not assume that $\tau$ is a stopping time
of the filtration $F$ generated by asset prices. We are instead interested in situations such that
the presence of an uncertain time-horizon induces some new uncertainty in the economy.

There are two sources of uncertainty related to optimal investment and consumption in the
presence of an uncertain time horizon, one stemming from the randomness of prices (market risk),
the other stemming from the randomness of the time of exit $\tau$ (timing risk). In general, these
two sources of uncertainty are not independent. Separating out these two sources of uncertainty
is a useful operation that may be achieved as follows. Conditioning up $F_t$ contains information
about risky asset prices up to time $t$, $P(\tau \leq t|F_t)$, for example, is the probability that the agent
has reached his time horizon at date $t$, given all possible information about asset prices. We
denote by $F_t = P(\tau \leq t|F_1)$, the conditional distribution function of timing uncertainty. Then $F_t$
is a $F$-submartingale with a Doob-Meyer decomposition $F_t = M_t + A_t$, where $M$ is a martingale
and $A$ is an increasing process. We further make the following assumption.

**Assumption 1.** $A$ is an absolutely continuous process with respect to Lebesgue measure, with
a bounded density denoted by $a$, i.e.,
\[ A_t = \int_0^t a_s ds. \]

There exists a positive constant $\varepsilon$, such that
\[ F_T \leq 1 - \varepsilon. \]

### 2.3 Preferences

The agent’s preferences are given by an utility function $u$, which is a twice continuously differ-
entiable function from $\mathbb{R}$ to $\mathbb{R}$. Moreover, we assume that:

**Assumption 2.** There exist two positive constants $\kappa$ and $N$, such that
\[ -N \leq u''(x) \leq -\kappa. \]
It should be pointed out that, a classical utility function is defined as a strictly increasing and strictly concave function. But in Assumption 2 the condition \( u''(x) \leq -\kappa \) is inconsistent with the increasing property.

However, \( u(x) \) must decrease near the infinity. We think this is not really a problem, and it is reasonable to assume the utility functions satisfy the previous Assumption 2. We have the following justifications: (1) The increasing utility function shows “more money more happiness”. But in the real world, a very huge number of money may not bring the owner happiness only, for example, the owner may worry about robbery and theft. So “more money more happiness” is doubtful. (2) In the finance model, it is possible to earn very huge money with a small endowment on a short time duration. But in practice, this is impossible. So we don’t care the property of \( u \) near the infinity.

The agent’s portfolio choice problem is to find an admissible strategy \( \pi \) which maximizes the expected utility of terminal wealth

\[
J(\pi(\cdot)) = \mathbb{E}(u(X^\pi_{\tau\wedge T}))
\]

over \( L^2_F(0, T; \mathbb{R}^n) \), where \( \tau \) is an agent uncertain time horizon. Using the definition of \( F_t = \mathbb{P}(\tau \leq t \mid \mathcal{F}_t) \) and its decomposition \( F_t = M_t + A_t \), we analysis

\[
J(\pi(\cdot)) = \mathbb{E} \left( \int_0^T a_s u(X^\pi_s) ds + (1 - F_T) u(X^\pi_T) \right).
\]

Because the stochastic integral item vanishes, we get the following expression:

\[
J(\pi(\cdot)) = \mathbb{E} \left( \int_0^T a_s u(X^\pi_s) ds + (1 - F_T) u(X^\pi_T) \right).
\]

We notice that the coefficients in (3) and (5) are \( \mathbb{F} \)-adapted.

3 Maximum principle

The stochastic maximum principle is one of the principal approaches in solving stochastic optimization problems. It gives a necessary condition that must be satisfied by any optimal solution, and a forward-backward stochastic differential equation (FBSDE) is involved in the necessary condition. In this section we are going to treat the optimal investment problem with uncertain time horizon by the maximum principle approach.

For the reader’s convenience, let us state here the maximum principle for general nonlinear systems, and then specialize to our optimal investment problem.

Given \( x \in \mathbb{R} \), we are to

Maximize \[ J(u(\cdot)) = \mathbb{E} \left[ \int_0^T l(t, x_t, u_t) dt + h(x_T) \right], \]

Subject to \[
\begin{cases} 
\quad dx_t = b(t, x_t, u_t) dt + \sigma(t, x_t, u_t) dW_t, \\
\quad x_0 = x.
\end{cases}
\]

Here, \( W(\cdot) \) is a \( n \)-dimensional Brownian motion; for any \( (x, u) \in \mathbb{R} \times \mathbb{R}^n \), \( b(\cdot, x, u), \sigma(\cdot, x, u), l(\cdot, x, u) \) are given \( \mathbb{F} \)-adapted square integrable processes and \( h(x) \) is a \( \mathcal{F}_T \)-measurable square integrable random variable respectively. Moreover, we assume that \( b, \sigma, l, h \) are continuously
differentiable in \((x, u)\), the partial derivatives \(b_x, b_u, \sigma_x, \sigma_u\) are bounded, and \(l_x, l_u, h_x\) are bounded by \(C(1 + |x| + |u|)\), where \(C\) is a constant. The set of admissible control is defined in the same way by \(L^2_{\mathcal{F}}(0, T; \mathbb{R}^n)\).

We define a Hamiltonian function as
\[
H(t, x, u, y, z) = l(t, x, u) + y \cdot b(t, x, u) + \sum_{j=1}^{n} z_j \cdot \sigma_j(t, x, u),
\] (8)
where \(\sigma = (\sigma_1, \ldots, \sigma_n)\) and \(z = (z_1, \ldots, z_n)\). The maximum principle asserts that if \((x^*(\cdot), u^*(\cdot))\) is optimal, then it must satisfy
\[
H_u(t, x^*_t, u^*_t, y_t, z_t) = 0,
\] (9)
where \((x^*(\cdot), y(\cdot), z(\cdot))\) is an \(\mathbb{F}\)-adapted solution to the following FBSDE
\[
\begin{align*}
\begin{cases}
&dx^*_t = H_y(t, x^*_t, u^*_t, y_t, z_t)dt + H_z(t, x^*_t, u^*_t, y_t, z_t)dW_t, \\
&-dy_t = H_x(t, x^*_t, u^*_t, y_t, z_t)dt - z_t dW_t, \\
&x^*_0 = x, \quad y_T = h_x(x_T).
\end{cases}
\end{align*}
\] (10)

We notice that in (10), the forward equation is just the state equation (7) with the optimal control \(u^*(\cdot)\). In stochastic optimal control theory, the backward equation in (10) is called the adjoint equation. The system consisting of the state equation, adjoint equation, and the maximum condition (9) is called a Hamiltonian system. For more details of maximum principle and Hamiltonian system theory, we refer to the book by Yong and Zhou and the reference therein.

Now we apply the above maximum principle to the optimal investment problem (3) and (4). Moreover, due to the presence of the concavity condition of \(u\) (see Assumption 2), the necessary condition obtained from the maximum principle is also a sufficient condition.

**Theorem 3.1.** Let Assumption 4 and Assumption 2 hold. The following two statements are equivalent:

(i) \((X^*(\cdot), \pi^*(\cdot))\) be an optimal pair for the problem (3) and (4);

(ii) There exists an \(\mathbb{F}\)-adapted process \(Y(\cdot)\), such that \((X^*(\cdot), \pi^*(\cdot), Y(\cdot))\) satisfying
\[
\begin{align*}
\begin{cases}
&dX^*_t = [r_t X^*_t + (\pi^*_t)^\top \theta_t]dt + (\pi^*_t)^\top dW_t, \\
&-dY_t = [r_t Y_t + a_t u'(X^*_t)]dt + \theta_t^\top Y_t dW_t, \\
&X^*_0 = x, \quad Y_T = (1 - F_T) u'(X^*_T).
\end{cases}
\end{align*}
\] (11)

**Proof.** Under Assumption 1 and Assumption 2, it is easy to check the conditions of the maximum principle are satisfied. Then, if \((X^*(\cdot), \pi^*(\cdot))\) is optimal, there must exist a pair of \(\mathbb{F}\)-adapted processes \((Y(\cdot), Z(\cdot))\), such that \((X^*(\cdot), \pi^*(\cdot), Y(\cdot), Z(\cdot))\) satisfying the following Hamiltonian system
\[
\begin{align*}
\begin{cases}
&0 = \theta_t Y_t + Z_t^\top, \\
&dX^*_t = [r_t X^*_t + (\pi^*_t)^\top \theta_t]dt + (\pi^*_t)^\top dW_t, \\
&-dY_t = [r_t Y_t + a_t u'(X^*_t)]dt - Z_t dW_t, \\
&X^*_0 = x, \quad Y_T = (1 - F_T) u'(X^*_T).
\end{cases}
\end{align*}
\]

Substituting the maximum condition \(Z_t = -\theta_t^\top Y_t\) into the forward and backward stochastic differential equations, we get (11).
On the other hand, we consider
\[
J(\pi^*(\cdot)) - J(\pi(\cdot)) = \mathbb{E} \int_0^T a_t(u(X^*_t) - u(X_t))dt + \mathbb{E} \left[ (1 - F_T)(u(X^*_T) - u(X_T)) \right].
\]
Since \( u \) is concave on \( x \),
\[
u(X^*) - u(X) \geq u'(X^*)(X^* - X), \quad \text{for any } X^*, X \in \mathbb{R}.
\]
Then,
\[
J(\pi^*(\cdot)) - J(\pi(\cdot)) \geq \mathbb{E} \int_0^T a_tu'(X^*_t)(X^*_t - X_t)dt + \mathbb{E} \left[ (1 - F_T)u'(X^*_T)(X^*_T - X_T) \right].
\]
If \((X^*(\cdot), \pi^*(\cdot), Y(\cdot))\) is a solution of (11). Applying Itô’s formula to \( Y(\cdot)X^*(\cdot) - X(\cdot) \) yields
\[
\mathbb{E} \left[ (1 - F_T)u'(X^*_T)(X^*_T - X_T) \right] = -\mathbb{E} \int_0^T a_tu'(X_t^*)(X_t^* - X_t)dt.
\]
We get \( J(\pi^*(\cdot)) - J(\pi(\cdot)) \geq 0 \), i.e., \((X^*(\cdot), \pi^*(\cdot))\) is optimal.

Theorem 3.1 shows the existence and uniqueness of the optimal strategy for Problem 3-5 are equivalent to the existence and uniqueness of the solution of FBSDE (11).

4 Forward-backward stochastic differential equations

FBSDE (11) is different from the conventional ones. Inspired by the generic type of BSDE, researchers focused on studying FBSDEs with the following classical type:
\[
\begin{aligned}
&dx_t = b(t, x_t, y_t, z_t)dt + \sigma(t, x_t, y_t, z_t)dW_t, \\
&-dy_t = f(t, x_t, y_t, z_t)dt - z_t dW_t, \\
x_0 = x, \quad y_T = \Phi(x_T),
\end{aligned}
\]
where, for any \((x, y, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{1 \times n}, b(\cdot, x, y, z), \sigma(\cdot, x, y, z), f(\cdot, x, y, z)\) are given \( \mathbb{F} \)-adapted square integrable processes and \( \Phi(x) \) is a \( \mathcal{F}_T \)-measurable square integrable random variable respectively (we refer to the book by Ma and Yong [5] and the reference therein). The backward equation in (12) is with the classical form, i.e., the diffusion coefficient is just \( z_t \), but this point is not kept in (11).

We would like to recall an existence and uniqueness result first obtained by Hu and Peng [5], and then generalized by Peng and Wu [11], for the classical FBSDE (12). This result will be used to get an existence and uniqueness result for the new type FBSDE (11). They adopt the notations
\[
p = (x, y, z), \quad A(t, p) = (-f(t, p), b(t, p), \sigma(t, p)),
\]
and assumed the following Lipschitz Assumption 3 and Monotonicity Assumption 4.

Assumption 3. \( A(t, p) \) and \( \Phi(x) \) are uniformly Lipschitz continuous with respect to \( p \) and \( x \) respectively.

Assumption 4. For a given constant \( G \in \{1, -1\} \), there exist three nonnegative constants \( \beta_1, \beta_2 \) and \( \nu_1 \) with \( \beta_1 + \beta_2 > 0, \nu_1 + \beta_2 > 0 \) such that
\[
G(A(t, p) - A(t, \bar{p}), p - \bar{p}) \leq -\beta_1 |x - \bar{x}|^2 - \beta_2 (|y - \bar{y}|^2 + |z - \bar{z}|^2),
\]
\[
G(\Phi(x) - \Phi(\bar{x}))(x - \bar{x}) \geq \nu_1 |x - \bar{x}|^2,
\]
for any \( p = (x, y, z), \bar{p} = (\bar{x}, \bar{y}, \bar{z}) \).
Under Assumption 3 and Assumption 4, they proved FBSDE (12) admits a unique solution \((x(\cdot), y(\cdot), z(\cdot)) \in L^2_\mathbb{F}(0,T;\mathbb{R}) \times L^2_\mathbb{F}(0,T;\mathbb{R}) \times L^2_\mathbb{F}(0,T;\mathbb{R}^{1 \times n})\).

Next, we will use an invertible transform to link the new type FBSDE (11) and the classical ones, then an existence and uniqueness result for FBSDE (11) is obtained. To this end, we need the following

**Assumption 5.** There exist two positive constants \(M\) and \(\lambda\), such that

\[2r_t + M \alpha_t - |\theta_t|^2 \geq \lambda, \quad \text{a.s.e.}\]

We notice that Assumption 5 is not strong. In the real financial market, it always holds true.

**Theorem 4.1.** Let Assumption 7, Assumption 2 and Assumption 3 hold. FBSDE (17) admits a unique solution \((X^*(\cdot), \pi^*(\cdot), Y(\cdot)) \in L^2_\mathbb{F}(0,T;\mathbb{R}) \times L^2_\mathbb{F}(0,T;\mathbb{R}^n) \times L^2_\mathbb{F}(0,T;\mathbb{R})\).

**Proof.** We introduce the following family of FBSDEs with the classical type parameterized by \(\alpha \in (-\infty, 0) \cup (0, +\infty)\).

\[
\begin{align*}
\frac{dx_t^\alpha}{dt} &= \left[ (r_t - |\theta|^2) x^\alpha_t + \frac{|\theta|^2}{\alpha} y^\alpha_t + \frac{1}{\alpha} z^\alpha_t \theta_t \right] dt + \left[ -\theta^\top_t x^\alpha_t + \frac{1}{\alpha} \theta^\top_t y^\alpha_t + \frac{1}{\alpha} z^\alpha_t \right] dW_t, \\
-dy^\alpha_t &= \left[ -\alpha(2r_t - |\theta|^2) x^\alpha_t + \alpha u'(x^\alpha_t) + (r_t - |\theta|^2) y^\alpha_t - \alpha \theta_t \right] dt - \alpha z^\alpha_t dW_t, \\
x^\alpha_0 &= x, \quad y^\alpha_T = \alpha x^\alpha_T + (1 - F_T) u'(x^\alpha_T).
\end{align*}
\]

(13)

We notice that if \((X^*(\cdot), \pi^*(\cdot), Y(\cdot))\) is a solution of FBSDE (11), then \((x^\alpha(\cdot), y^\alpha(\cdot), z^\alpha(\cdot))\) solves FBSDE (13), where

\[x^\alpha_t = X^*_t, \quad y^\alpha_t = Y_t + \alpha X^*_t, \quad z^\alpha_t = \alpha (\pi^*_t - \theta^\top_t Y_t), \quad t \in [0, T].\]

The above transformation is invertible. If \((x^\alpha(\cdot), y^\alpha(\cdot), z^\alpha(\cdot))\) is a solution of FBSDE (13), then

\[X^*_t = x^\alpha_t, \quad Y_t = y^\alpha_t - \alpha x^\alpha_t, \quad \pi^*_t = \frac{1}{\alpha} (z^\alpha_t)^\top + \frac{1}{\alpha} \theta^\top_t y^\alpha_t - \theta^\top_t x^\alpha_t, \quad t \in [t, T],\]

solves FBSDE (11). In other words, the existence and uniqueness of FBSDE (11) and FBSDEs (13) for each \(\alpha \neq 0\) are equivalent. Then if we can find a special \(\alpha\) such that FBSDE (13) admits a unique solution, then the same conclusion holds for FBSDE (13) with other \(\alpha\) and for FBSDE (11).

Now we check the coefficients of FBSDE (13) with some \(\alpha\) satisfies Assumption 3 and Assumption 4. Because \(u''\) and \(f\) is bounded, Assumption 3 holds true for any \(\alpha \neq 0\). Now, for a given constant \(G \in \{1, -1\}\), for any \(p = (x, y, z), \bar{p} = (\bar{x}, \bar{y}, \bar{z}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{1 \times n}\), we denote \(\hat{x} = x - \bar{x}, \hat{y} = y - \bar{y}, \hat{z} = z - \bar{z}\), \(\hat{p} = (\hat{x}, \hat{y}, \hat{z})\) and calculate

\[G(A(t, p) - A(t, \bar{p}))(p - \bar{p}) = G \left[ \alpha(2r_t - |\theta|^2) \hat{x}^2 - a_t \left( u'(x) - u'(\bar{x}) \right) \hat{x} + \frac{1}{\alpha} |\theta^\top_t \hat{y} + \hat{z}|^2 \right] \]

\[= G \left[ \alpha(2r_t - |\theta|^2) \hat{x}^2 - a_t \left( \int_0^1 u''(\bar{x} + h\hat{x}) dh \right) \hat{x}^2 + \frac{1}{\alpha} |\theta^\top_t \hat{y} + \hat{z}|^2 \right] \]

\[= G \left[ \alpha(2r_t - |\theta|^2) \hat{x}^2 - a_t \left( \int_0^1 u''(\bar{x} + h\hat{x}) dh \right) \hat{x}^2 + \frac{G}{\alpha} |\theta^\top_t \hat{y} + \hat{z}|^2 \right] \]

\[= G(\Phi(x) - \Phi(\bar{x}))(x - \bar{x}) = G \left[ (1 - F_T) \left( u'(x) - u'(\bar{x}) \right) \hat{x} + \alpha \hat{x}^2 \right] \]

\[= G \left[ (1 - F_T) \left( \int_0^1 u''(\bar{x} + h\hat{x}) dh \right) + \alpha \right] \hat{x}^2.\]
In order to ensure \( (G/\alpha)|\theta_t^T \hat{y} + \hat{z}|^2 \leq 0 \), we must select \( G \) and \( \alpha \) with different sign, i.e., \( G = 1, \alpha < 0 \) or \( G = -1, \alpha > 0 \). But, if we select the first case \( G = 1, \alpha < 0 \), by the concavity of \( u \), we have \( G(\Phi(x) - \Phi(\bar{x}))(x - \bar{x}) \leq 0 \), thus it is impossible Assumption 4 holds true. So, in the following, we select \( G = -1, \alpha > 0 \). We first consider the expression of \(- (\Phi(x) - \Phi(\bar{x}))(x - \bar{x})\).

\[
-(\Phi(x) - \Phi(\bar{x}))(x - \bar{x}) = -(1 - F_T) \left( \int_0^1 u''(\bar{x} + h\hat{x}) dh \right) - \alpha \hat{x}^2 \geq (\varepsilon \kappa - \alpha)\hat{x}^2.
\]

So we need select \( 0 < \alpha < \varepsilon \kappa \). Secondly, we consider the expression of \(- (A(t, p) - A(t, \bar{p}), p - \bar{p})\). By Assumption 5 and the further restriction on \( \alpha: 0 < \alpha < \min(\varepsilon \kappa, \kappa/M) \), we get

\[
-A(t, p) - A(t, \bar{p}), p - \bar{p}) \leq -\alpha(2r_t - |\theta_t|^2) + a_t \left( \int_0^1 u''(\bar{x} + h\hat{x}) dh \right) \hat{x}^2 \leq -\alpha(2r_t - |\theta_t|^2 - \kappa a_t) \hat{x}^2 \leq -\alpha(2r_t + Ma_t - |\theta_t|^2) \hat{x}^2 \leq -\alpha \lambda \hat{x}^2.
\]

So Assumption 4 holds true when \( 0 < \alpha < \min(\varepsilon \kappa, \kappa/M) \). Then FBSDE 11 has a unique solution and we complete the proof. \( \square \)

Combining with Theorem 3.1 and Theorem 4.1

**Theorem 4.2.** Under Assumption 7, Assumption 2 and Assumption 4, the optimal investment problem 3 and 4 admits a unique optimal pair \((X^*(\cdot), \pi^*(\cdot)) \in L^2(0,T;\mathbb{R}) \times L^2(0,T;\mathbb{R}^n)\) determined by FBSDE 11.

**References**


