Suitable solutions for the Navier–Stokes problem with an homogeneous initial value.

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Abstract :

This paper is devoted to the study of strong or weak solutions of the Navier–Stokes equations in the case of an homogeneous initial data. The case of small initial data is discussed. For large initial data, an approximation is developed, in the spirit of a paper of Vishik and Fursikov. Qualitative convergence is obtained by use of the theory of Muckenhoupt weights.

Keywords :

Navier–Stokes equations; Morrey–Campanato spaces; Muckenhoupt weights; maximal regularity; self–similar solutions

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Introduction.

In this paper, we shall study the Cauchy problem for an incompressible 3D Navier–Stokes problem with no boundary and no external force

(1) $$\begin{cases} \partial_t\vec{u} + \vec{u}.\vec{\nabla}\vec{u} = \Delta\vec{u} - \vec{\nabla}p \\ \vec{u}_{|t=0} = \vec{u}_0 \\ \vec{\nabla}.\vec{u} = 0 \end{cases}$$

where $\vec{u}(t,x)$ is a time-dependent divergence-free vector field on $\mathbb{R}^3$ ($t>0$, $x\in\mathbb{R}^3$) and where the initial value $\vec{u}_0$ is homogeneous :

(2) $$\text{for } \lambda>0, \quad \lambda\vec{u}_0(\lambda x) = \vec{u}_0(x).$$

The homogeneity condition on $\vec{u}_0$ fits the scaling property of the Navier-Stokes equations : if $\vec{u}$ is a solution of the Cauchy problem with initial value, then $\vec{u}_\lambda$ defined by $\vec{u}_\lambda(t,x) = \lambda\vec{u}(\lambda^2 t, \lambda x)$ (where $\lambda>0$) is a solution for the Cauchy problem with initial value $\lambda\vec{u}_0(\lambda x)$.

Of course, we aim to exhibit self–similar solutions ($\lambda\vec{u}(\lambda^2 t,\lambda x) = \vec{u}(t,x)$), but up to now this can be done only for small initial values. For such small values, the formalism of mild solutions [KAT 84] [CAN 95], based on Banach's contraction principle, provides solutions together with some uniqueness which grants self–similarity. When we deal with large initial values, the formalism of mild solutions breaks down and we can only exhibit weak solutions, through a compactness argument based on some energy estimates. For such solutions, we have no uniqueness, so that we cannot conclude for self–similarity. Those energy estimates cannot be a direct consequence of Leray's theory [LER 34], since, when $\vec{u}_0 \neq 0$, homogeneity implies that $\|\vec{u}_0\|_2 = +\infty$. We thus have to replace Leray's energy inequality by Scheffer's local energy inequality [SCH 77]. We shall describe some consequences of this inequality, through the use of Caffarelli, Kohn and Nirenberg's regularity criterion [CAF 82].

Weak solutions are usually obtained through mollification [LEM 02] or truncation [LEM 99] [BAS 06]. In the last sections, we shall describe another approximation of the Navier–Stokes equation which provide suitable solutions and preserves the scaling property of the equations. Those approximations are modifications of a model studied by Vishik and Fursikov [VIS 77].

1. Suitable solutions.

In this section, we recall previous results from [LEM 99], [LEM 02] and [LEM 07] on weak solutions for the Navier–Stokes equations.

In Leray's theory [LER 34], a *weak solution* of equations (1) is a solution $\vec{u} \in L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x$ defined on $(0,+\infty)\times\mathbb{R}^3$ which satisfies the energy inequality

(3) $$\|\vec{u}(t,.)\|_2^2 + 2\int_0^t \|\vec{\nabla}\otimes\vec{u}\|_2^2\, ds \le \|\vec{u}_0\|_2^2$$

where the initial value $\vec{u}_0$ is a square-integrable divergence-free vector field. In that case, the pressure $p(t,x)$ belongs to $L^2_t L^{3/2}_x$ and can be recovered from $\vec{u}$ by the formula

$$p = -\sum_{i=1}^{3}\sum_{j=1}^{3} \frac{1}{\Delta}\partial_i\partial_j(u_i u_j). \tag{4}$$

In particular, a Leray solution $\vec{u}$ belongs to $L^{8/3}_t L^4_x$. If we assume more regularity on the solution $\vec{u}$ ($\vec{u} \in L^4_t L^4_x$), then the inequality (3) becomes an equality. Indeed, in that case $p \in L^2_t L^2_x$ and thus $\partial_t \vec{u} \in L^2_t \dot{H}^{-1}_x$ (rewriting $\vec{u}.\vec{\nabla}\vec{u}$ as $\vec{\nabla}.(\vec{u}\otimes\vec{u})$). Thus, we have

$$\frac{d}{dt}\|\vec{u}(t,.)\|_2^2 = 2\langle \partial_t \vec{u}(t,.)|\vec{u}(t,.)\rangle_{\dot{H}^{-1},\dot{H}^1} = -2\int |\vec{\nabla}\otimes\vec{u}|^2\, dx - 2\int \vec{u}.(\vec{u}.\vec{\nabla})\vec{u}\, dx - 2\int \vec{u}.\vec{\nabla}p\, dx \tag{5}$$

where

$$\int \vec{u}.\vec{\nabla}p\, dx = -\int p\vec{\nabla}.\vec{u}\, dx = 0 \tag{6}$$

and

$$\int \vec{u}.(\vec{u}.\vec{\nabla})\vec{u}\, dx = -\int \vec{u}.(\vec{u}.\vec{\nabla})\vec{u}\, dx - \int |\vec{u}|^2\vec{\nabla}.\vec{u}\, dx = -\frac{1}{2}\int |\vec{u}|^2\vec{\nabla}.\vec{u}\, dx = 0 \tag{7}$$

since $\vec{u}$ is divergence free. The integrations by part involved in (5), (6) and (7) may be enlighted by describing the distribution $\partial_t|\vec{u}|^2 + 2|\vec{\nabla}\otimes\vec{u}|^2$ as a divergence :

$$\partial_t|\vec{u}|^2 + 2|\vec{\nabla}\otimes\vec{u}|^2 = \sum_{i=1}^{3}\partial_i(2\vec{u}.\partial_i\vec{u} - (|\vec{u}|^2 + 2p)u_i) + R \tag{8}$$

where

$$R = (|\vec{u}|^2 + 2p)\ \vec{\nabla}.\vec{u} = 0 \tag{9}$$

When $\vec{u}$ is a Leray solution but does not belong to $L^4_t L^4_x$, we cannot write $\partial_t|\vec{u}|^2 = 2\partial_t\vec{u}.\vec{u}$. Energy equality is not fullfilled (or, at least, is not known to be fullfilled). An usual way to exhibit Leray solutions consists in mollyfying the nonlinearity : we take a nonnegative $\omega \in \mathcal{D}(\mathbb{R}^3)$ such that $\int \omega\, dx = 1$, we define, for $\epsilon > 0$, $\omega_\epsilon(x) = \epsilon^{-3}\omega(\epsilon^{-1}x)$ and we change equations (1) into

$$\begin{cases} \partial_t\vec{u}_\epsilon + (\omega_\epsilon * \vec{u}_\epsilon).\vec{\nabla}\vec{u}_\epsilon = \Delta\vec{u}_\epsilon - \vec{\nabla}p_\epsilon \\ \vec{u}_{\epsilon|t=0} = \vec{u}_0 \\ \vec{\nabla}.\vec{u}_\epsilon = 0 \end{cases} \tag{10}$$

and we find a solution $\vec{u}_\epsilon \in L^\infty_t L^2_x \cap L^2_t\dot{H}^1_x$ such that $\partial_t\vec{u}_\epsilon \in L^2_t\dot{H}^{-1}_x$. We get that

$$\partial_t|\vec{u}_\epsilon|^2 + 2|\vec{\nabla}\otimes\vec{u}_\epsilon|^2 = \sum_{i=1}^{3}\partial_i(2\vec{u}_\epsilon.\partial_i\vec{u}_\epsilon - |\vec{u}_\epsilon|^2\omega_\epsilon * u_{\epsilon,i} - 2p_\epsilon u_{\epsilon,i}) + R_\epsilon \tag{11}$$

where

$$R_\epsilon = |\vec{u}_\epsilon|^2\ \omega_\epsilon * (\vec{\nabla}.\vec{u}_\epsilon) + 2p_\epsilon\ \vec{\nabla}.\vec{u}_\epsilon = 0 \tag{12}$$

By a compactness argument based on Rellich's theorem (for details, we refer to [LEM 02] chapters 13 and 14), there is a sequence $\epsilon_k \to 0$ and a distribution $\vec{u} \in L^\infty_t L^2 \cap L^2_t\dot{H}^1_x$ such that $\vec{u}_{\epsilon_k}$ converges to $\vec{u}$ weakly in $L^2_t\dot{H}^1_x$ and strongly in L^2 norm on every compact subset of $(0,+\infty)\times\mathbb{R}^3$. Thus, we have (in $\mathcal{D}'((0,+\infty)\times\mathbb{R}^3)$)

$$\lim_{\epsilon_k\to 0}\partial_t\vec{u}_{\epsilon_k} + (\omega_{\epsilon_k} * \vec{u}_{\epsilon_k}).\vec{\nabla}\vec{u}_{\epsilon_k} - \Delta\vec{u}_{\epsilon_k} + \vec{\nabla}p_{\epsilon_k} = \partial_t\vec{u} + \vec{u}.\vec{\nabla}\vec{u} - \Delta\vec{u} + \vec{\nabla}p = 0 \tag{13}$$

and

$$(14)\quad \lim_{\epsilon_k\to 0}\partial_t|\vec u_{\epsilon_k}|^2+\sum_{i=1}^3\partial_i(-2\vec u_{\epsilon_k}.\partial_i\vec u_{\epsilon_k}+|\vec u_{\epsilon_k}|^2\omega_{\epsilon_k}*u_{\epsilon_k,i}+2p_{\epsilon_k}u_{\epsilon_k,i})=\partial_t|\vec u|^2+\sum_{i=1}^3\partial_i(-2\vec u.\partial_i\vec u+|\vec u|^2u_i+2pu_i)$$

However, there is no reason that $|\vec\nabla\otimes\vec u_{\epsilon_k}|^2$ should converge to $|\vec\nabla\otimes\vec u|^2$. The best we can get is

$$(15)\qquad \lim_{\epsilon_k\to 0}|\vec\nabla\otimes\vec u_{\epsilon_k}|^2=|\vec\nabla\otimes\vec u|^2+\mu$$

where μ is a non-negative distribution on $(0,+\infty)\times\mathbb{R}^3$ (hence a locally finite non-negative measure). This gives

$$(16)\qquad \partial_t|\vec u|^2+2|\vec\nabla\otimes\vec u|^2=\Delta|\vec u|^2-\vec\nabla.((|\vec u|^2+2p)\vec u)-\mu$$

This is Scheffer's local energy inequality [SCH 77].

In contrast with Leray's inequality (3), we don't need that $\vec u$ be square-integrable in inequality (16). Solutions of the Navier-Stokes equations which satisfy (16) will be called suitable (following [CAF 82]) :

Definition 1

A time-dependent divergence-free vector field $\vec u$ defined on $(0,T)\times\mathbb{R}^3$ will be a suitable solution of the Navier–Stokes equations if
i) $\vec u$ belongs locally (in time and space) to $L^\infty_tL^2_x\cap L^2_t\dot H^1_x$
ii) there exists a distribution $p\in\mathcal{D}'((0,T)\times\mathbb{R}^3)$ such that $\partial_t\vec u+\vec u.\vec\nabla\vec u-\Delta\vec u+\vec\nabla p=0$
iii) locally in time and space, $\vec u$ belongs to $L^3_tL^3_x$ and p belongs to $L^{3/2}_tL^{3/2}_x$
iv) $\vec u$ satisfies Scheffer's inequality : there exists a locally finite non-negative measure μ on $(0,T)\times\mathbb{R}^3$ such that $\partial_t|\vec u|^2+2|\vec\nabla\otimes\vec u|^2=\Delta|\vec u|^2-\vec\nabla.((|\vec u|^2+2p)\vec u)-\mu$.

The main interest of suitable solutions is the regularity criterion of Caffarelli, Kohn and Nirenberg [CAF 82] :

Theorem 1 :

There exists two constants $\epsilon_0>0$ and $C_0>0$ such that if $T>0$, if $x_0\in\mathbb{R}^3$, if $0<r^2<t_0<T$, if $0<\epsilon<\epsilon_0$, if $\vec u$ is a suitable solution of the Navier–Stokes equations on $(0,T)\times\mathbb{R}^3$ such that

$$(17)\qquad \int\int_{|x-x_0|<r,t_0-r^2<t<t_0}|\vec u(t,x)|^3+|p(t,x)|^{3/2}\,dx\,dt<\epsilon r^2$$

then

$$(18)\qquad \sup_{|x-x_0|<r/2,t_0-r^2/4<t<t_0}|\vec u(t,x)|<C_0\epsilon^{1/3}r^{-1}$$

Inequality (16) is a key tool to develop a theory of weak solutions for initial values $\vec u_0$ with infinite energy ($\|\vec u_0\|_2=+\infty$). In [LEM 99] [LEM 02] a theory has been developed to exhibit suitable weak solutions associated to an initial value $\vec u_0$ which is uniformly locally square integrable ($\sup_{x_0\in\mathbb{R}^3}\int_{|x-x_0|\leq 1}|\vec u_0(x)|^2\,dx<+\infty$). The basic idea of the proof is to consider the mollified equations (10) and to compute the L^2_{uloc} norm of $\vec u_\epsilon$ as

$$(19)\qquad \|\vec u_\epsilon\|_{L^2_{uloc}}=\sup_{x_0\in\mathbb{R}^3}\|\varphi_0(x-x_0)\vec u_\epsilon\|_2$$

for some $\varphi_0\in\mathcal{D}(\mathbb{R}^3)$ (with $\varphi_0\neq 0$). In contrast with the finite-energy case, p_ϵ cannot be computed as $p_\epsilon=-\sum_{i=1}^3\sum_{j=1}^3\frac{1}{\Delta}\partial_i\partial_j(\omega_\epsilon*u_{\epsilon,i}\,u_{\epsilon,j})$ since the kernel of the convolution operator $\frac{1}{\Delta}\partial_i\partial_j$ has slow decay at infinity, hence is not defined on L^p_{uloc}. But p_ϵ is well defined up to a constant additive term, so that $\vec\nabla p_\epsilon$ is well defined : the kernel of $\partial_k\frac{1}{\Delta}\partial_i\partial_j$ has enough decay at infinity to operate on L^p_{uloc}. Then formulas (11) and (12) remain true. Carefully integrated against test functions $\varphi(x)=\varphi_0^2(x-x_0)$, they give a control independent of ϵ : we start from the identity

$$(20)\qquad \int\varphi(x)|\vec u_\epsilon(t,x)|^2\,dx+2\int_0^t\int\varphi(x)|\vec\nabla\otimes\vec u(s,x)|^2\,dx\,dt=\int\varphi(x)|\vec u_0(x)|^2\,dx+I_\epsilon(t)$$

with

$$(21)\qquad I_\epsilon(t) = \int\int_0^t |\vec u_\epsilon(s,x)|^2 \Delta\varphi(x)\,dx\,ds + \int_0^t \int |\vec u_\epsilon|^2 (\omega_\epsilon * \vec u_\epsilon) * \vec\nabla\varphi\,dx\,ds + 2\int\int_0^t p_\epsilon \vec u_\epsilon.\vec\nabla\varphi\,dx\,ds$$

and defining

$$(22)\qquad \alpha_\epsilon(t) = \sup_{x_0\in\mathbb{R}^3} \int \varphi_0^2(x-x_0)|\vec u_\epsilon(t,x)|^2\,dx \text{ and } \beta_\epsilon(t) = \sup_{x_0\in\mathbb{R}^3}\int_0^t\int \varphi_0^2(x-x_0)|\vec\nabla\otimes\vec u(s,x)|^2\,dx\,dt$$

we find that

$$(23)\qquad I_\epsilon(t) \le C(\int_0^t \alpha_\epsilon(s)ds + (\int_0^t \alpha_\epsilon^3(s)\,ds)^{1/4}(\beta_\epsilon(t) + \int_0^t \alpha_\epsilon(s)\,ds)^{3/4})$$

In [LEM 02], we show that inequalities (20) and (23) provide a control uniform in ϵ on a time interval $(0,T)$ with $T = O(\min(1, \|\vec u_0\|_{L^2_{uloc}}^{-2})$. Then the same compactness argument as in the case of finite-energy initial values allows us to show that :

Theorem 2

Let $\vec u_0 \in (L^2_{uloc}(\mathbb{R}^3))^3$ be such that $\vec\nabla.\vec u_0 = 0$. Then, there exists a positive constant C_0 (which does not depend on $\vec u_0$) such that, defining $T_0 = \frac{1}{C_0^4 \sup(1, \|\vec u_0\|^2_{L^2_{uloc}})}$, the equations (1) have a suitable solution $\vec u$ on $(0,T_0)\times\mathbb{R}^3$ such that for all $0<t<T_0$ we have

$$(24)\qquad \|\vec u(t,.)\|_{L^2_{uloc}} \le \sqrt{C_0}\|\vec u_0\|_{L^2_{uloc}}(1-\frac{t}{T_0})^{-1/4}$$

and

$$(25)\qquad \sup_{x_0\in\mathbb{R}^3}\int_0^t\int_{|x-x_0|<1} |\vec\nabla\otimes\vec u(s,x)|^2\,dx\,ds \le C_0\|\vec u_0\|^2_{L^2_{uloc}}(1-\frac{t}{T_0})^{-1/2}.$$

Moreover, the decay rate, when x goes to infinity, may be controlled [LEM 02] :

Theorem 3

Let $\vec u_0 \in (L^2_{uloc}(\mathbb{R}^3))^3$ be such that $\vec\nabla.\vec u_0 = 0$. For some positive T^*, let $\vec u$ be a suitable solution for the Navier–Stokes problem on $(0,T^*)\times\mathbb{R}^3$ with initial value $\vec u_0$ (such that p is given by $\vec\nabla p = -\sum_{i=1}^3\sum_{j=1}^3 \vec\nabla\frac{1}{\Delta}\partial_i\partial_j(u_iu_j)$). Let $\theta\in\mathcal{D}(\mathbb{R}^3)$ be equal to 1 on a neighbourhood of 0. For $R>0$, let $\chi_R(x) = 1-\theta(\frac{x}{R})$. Then, for every $T\in(0,T^*)$, there exists a positive constant C_T such that, for all $0<t<T$ and all $R>1$, we have

$$(26)\qquad \|\chi_R\vec u(t,.)\|_{L^2_{uloc}} \le \sqrt{C_T}\,(\|\chi_R\vec u_0\|_{L^2_{uloc}} + \sqrt{\frac{1+\ln R}{R}})$$

and

$$(27)\qquad \sup_{x_0\in\mathbb{R}^3}\int_0^t\int_{|x-x_0|<1}\chi_R(x)|\vec\nabla\otimes\vec u(s,x)|^2\,dx\,ds \le C_T\,(\|\chi_R\vec u_0\|^2_{L^2_{uloc}} + \frac{1+\ln R}{R})$$

The constant C_T depends only on T, $\sup_{0<t<T}\|\vec u(t,.)\|_{L^2_{uloc}}$ and $\sup_{x_0\in\mathbb{R}^3}\int_0^T\int_{\|x-x_0\|<1}|\vec\nabla\otimes\vec u(s,x))|^2\,dx\,ds$.

Now, if we want to study the Navier–Stokes equations with an initial value $\vec u_0$ which is homogeneous (as given by (2)) and uniformly locally square-integrable, this initial value will belong to a Morrey–Campanato space :

Definition 2 :

For $1<p\le q<\infty$, the homogeneous Morrey–Campanato space $\dot M^{p,q}(\mathbb{R}^3)$ is defined as the space of locally p-integrable functions f such that

$$(28)\qquad \sup_{x_0\in\mathbb{R}^3}\sup_{0<R<\infty} R^{3(1/q-1/p)}(\int_{|x-x_0|<R}|f(x)|^p\,dx)^{1/p} < \infty.$$

or equivalently

$$\sup_{R>0} R^{3/q} \|f(Rx)\|_{L^p_{uloc}} < +\infty \tag{29}$$

A direct consequence of (29) is that, when $\vec u_0 \in \dot M^{2,3}$ the Proof of Theorem 2 can be adapted to any scale, hence will provide a solution on any time interval $(0,T)$, anf finally (through a diagonal extraction process) a global solution [LEM 07] :

Theorem 4

Let $\vec u_0 \in (\dot M^{2,3}(\mathbb{R}^3))^3$ be such that $\vec\nabla.\vec u_0 = 0$. Then, there exists a positive constant C_0 (which does not depend on $\vec u_0$) such that, defining $T_0 = \frac{1}{C_0^4 \sup(1, \|\vec u_0\|^2_{\dot M^{2,3}})}$, equations (1) have a suitable solution $\vec u$ on $(0,+\infty)\times\mathbb{R}^3$ such that

$$\sup_{x_0\in\mathbb{R}^3,\ R>0,\ t>0} \frac{1}{R+\sqrt{\frac{t}{T_0}}} \int_{|x-x_0|<R} |\vec u(t,x)|^2\ dx \le C_0 \|\vec u_0\|^2_{\dot M^{2,3}} \tag{30}$$

and

$$\sup_{x_0\in\mathbb{R}^3,\ t>0} \sqrt{\frac{T_0}{t}} \int_0^t \int_{|x-x_0|<\sqrt{\frac{t}{T_0}}} |\vec\nabla\otimes\vec u(s,x)|^2\ dx\ ds \le C_0\|\vec u_0\|^2_{\dot M^{2,3}}. \tag{31}$$

2. Small solutions.

When $\vec u_0$ is small, we have many further results on the solutions of equations (1) :

Theorem 5 :

Let $\vec u_0 \in (\dot M^{2,3}(\mathbb{R}^3))^3$ be such that $\vec\nabla.\vec u_0 = 0$. Then, there exist constants C_0, C_1 and ϵ_0 (which don't depend on $\vec u_0$) such that, if $\|\vec u_0\|_{\dot M^{2,3}} < \epsilon_0$, the following assertions are true :
(A) [Existence] equations (1) have a suitable solution $\vec u$ on $(0,+\infty)\times\mathbb{R}^3$ (with pressure p given by $\vec\nabla p = -\sum_{i=1}^3\sum_{j=1}^3 \vec\nabla \frac{1}{\Delta}\partial_i\partial_j(u_iu_j)$) such that

$$\sup_{x_0\in\mathbb{R}^3,\ t>0,\ t>s>0} \frac{1}{\sqrt t} \int_{|x-x_0|<\sqrt t} |\vec u(s,x)|^2\ dx \le C_0\|\vec u_0\|^2_{\dot M^{2,3}} \tag{32}$$

and

$$\sup_{x_0\in\mathbb{R}^3,\ t>0} \sqrt{\frac{1}{t}} \int_0^t\int_{|x-x_0|<\sqrt t} |\vec\nabla\otimes\vec u(s,x)|^2\ dx\ ds \le C_0\|\vec u_0\|^2_{\dot M^{2,3}}. \tag{33}$$

(B) [Uniqueness] If $\vec u$ and $\vec v$ are two suitable solutions of (1) which satisfy (32) and (33) then $\vec u = \vec v$.
(C) [Regularity] The solution $\vec u$ satisfies

$$\sup_{t>0}\sqrt t\|\vec u(t,.)\|_\infty + \sup_{t>0}\|\vec u(t,.)\|_{\dot M^{2,3}} \le C_1\|\vec u\|_{\dot M^{2,3}} \tag{34}$$

(D) [Convergence] If $\vec u_\epsilon$ is the solution of the mollified equations (10), then $\vec u_\epsilon$ converge to $\vec u$ in $\mathcal{D}'((0,+\infty)\times\mathbb{R}^3)$ as ϵ goes to 0. If moreover $\vec u_0 \in (E_2)^3$ where

$$f\in E_2 \Leftrightarrow f\in L^2_{uloc} \text{ and } \lim_{x_0\to\infty}\int_{|x-x_0|<1} |f(x)|^2\ dx = 0 \tag{35}$$

then we have, for all $T>0$

$$\lim_{\epsilon\to 0}\sup_{0<t<T} \|\vec u(t,.)-\vec u_\epsilon(t,.)\|_{L^2_{uloc}} = 0 \tag{36}$$

(E) [Self–similarity] If $\vec{u}_0$ is homogeneous ($\lambda\vec{u}_0(\lambda x) = \vec{u}_0(x)$ for $\lambda > 0$) then $\vec{u}$ is self–similar ($\lambda\vec{u}(\lambda^2 t, \lambda x) = \vec{u}(t,x)$).

Proof : Let $\vec{u}_0 \in (\dot{M}^{2,3}(\mathbb{R}^3))^3$ be such that $\vec{\nabla}.\vec{u}_0 = 0$. If $\|\vec{u}_0\|_{\dot{M}^{2,3}} < 1$, (30) and (31) in Theorem 4 give us (32) and (33). (32) and (33) give us that for some constant C_2

$$\sup_{x_0\in\mathbb{R}^3,\ t>0} \frac{1}{t}\int_0^t \int_{|x-x_0|<\sqrt{t}} |\vec{u}(s,x)|^3\ dx\ ds \le C_2\|\vec{u}_0\|^3_{\dot{M}^{2,3}}. \tag{37}$$

Moreover Lemma 32.2 in [LEM 02] shows us that, for given $x_0 \in \mathbb{R}^3$ and $t > 0$, we may modify p on $B(x_0,\sqrt{t})\times(0,t)$ such that

$$\frac{1}{t}\int_0^t \int_{|x-x_0|<\sqrt{t}} |p(s,x)|^{3/2}\ dx\ ds \le C_2\|\vec{u}_0\|^3_{\dot{M}^{2,3}}. \tag{38}$$

Thus, if $\vec{u}_0$ is small enough, we are allowed to use Theorem 1 and to find that, for almost $|x - x_0| < \sqrt{t}/2$ and almost $3t/4 < s < t$ we have $|\vec{u}(s,x)| \le C_1\|\vec{u}_0\|_{\dot{M}^{2,3}}\frac{1}{\sqrt{t}}$. This gives that

$$\sup_{0<t} \sqrt{t}\|\vec{u}(t,.)\|_\infty \le C_1\|\vec{u}_0\|_{\dot{M}^{2,3}} \tag{39}$$

Now, if we want to estimate $\frac{1}{R}\int_{|x-x_0|<R} |\vec{u}(t,x)|^2\ dx$, we just use (32) if $t < R^2$ and (39) if $t \ge R^2$. Thus, we get (34).

We now consider the uniqueness. Let $(\vec{u},p)$ and $(\vec{v},q)$ be two suitable solutions of (1) which satisfy estimates described in (A). We get rid of the pressure terms by using the Leray projection operator $\mathbb{P}$ on solenoidal vector fields. We define a bilinear operator B as

$$B(\vec{f},\vec{g}) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{f}\otimes\vec{g})\ ds \tag{40}$$

The following estimates are classical and easy to prove [LEM 02] for $T \in (0,+\infty)$ (with a constant C_0 which does not depend on T)

$$\sup_{0<t<T} \|B(\vec{f},\vec{g})\|_{L^2_{uloc}} + \|B(\vec{g},\vec{f})\|_{L^2_{uloc}} \le C_0 \sup_{0<t<T}\|\vec{f}(t,.)\|_{L^2_{uloc}} \sup_{0<t<T}\sqrt{t}\|\vec{g}(t,.)\|_\infty \tag{41}$$

$$\sup_{0<t<T} \|B(\vec{f},\vec{g})\|_{\dot{M}^{2,3}} + \|B(\vec{g},\vec{f})\|_{\dot{M}^{2,3}} \le C_0 \sup_{0<t<T}\|\vec{f}(t,.)\|_{\dot{M}^{2,3}} \sup_{0<t<T}\sqrt{t}\|\vec{g}(t,.)\|_\infty \tag{42}$$

and

$$\sup_{0<t<T}\sqrt{t}\|B(\vec{f},\vec{g})\|_\infty \le C_0(\sup_{0<t<T}\|\vec{f}(t,.)\|_{\dot{M}^{2,3}} \sup_{0<t<T}\sqrt{t}\|\vec{g}(t,.)\|_\infty + \sup_{0<t<T}\|\vec{g}(t,.)\|_{\dot{M}^{2,3}} \sup_{0<t<T}\sqrt{t}\|\vec{f}(t,.)\|_\infty) \tag{43}$$

We go back to $\vec{u}$ and $\vec{v}$. We define $\vec{w} = \vec{u} - \vec{v}$. We have

$$\vec{u} = e^{t\Delta}\vec{u}_0 - B(\vec{u},\vec{u}) \text{ and } \vec{v} = e^{t\Delta}\vec{u}_0 - B(\vec{v},\vec{v}) \tag{44}$$

hence

$$\vec{w} = B(\vec{v},\vec{v}) - B(\vec{u},\vec{u}) = -B(\vec{w},\vec{v}) - B(\vec{u},\vec{w}) \tag{45}$$

Combining (41) and (39) we find that

$$\sup_{0<t}\|\vec{w}(t,.)\|_{L^2_{uloc}} \le 2C_0C_1\|\vec{u}_0\|_{\dot{M}^{2,3}} \sup_{0<t}\|\vec{w}(t,.)\|_{L^2_{uloc}} \tag{45}$$

so that $\vec{w} = 0$ if $2C_0C_1\|\vec{u}_0\|_{\dot{M}^{2,3}} < 1$.

Uniqueness implies self-similarity when $\vec{u}_0$ is homogeneous : if $\vec{u}_0$ is homogeneous and $\vec{u}$ a solution of (1) which satisfies (32) and (33), then $\lambda\vec{u}(\lambda^2 t,\lambda x)$ is still a solution of (1) which satisfies (32) and (33); by uniqueness, we get $\vec{u}(t,x) = \lambda\vec{u}(\lambda^2 t,\lambda x)$.

We now prove point (D). The set $(\vec u_\epsilon)_{\epsilon>0}$ is a relatively compact subset of $(\mathcal{D}'(\mathbb{R}^3))^3$. If $\vec u$ is the limit of some sequence $\vec u_{\epsilon_k}$ with $\epsilon_k\to 0$, then $\vec u$ will be a solution of (1) and satisfy (32) and (33); but such a solution has been seen being unique. Thus, any sequence $\vec u_{\epsilon_k}$ with $\epsilon_k\to 0$ will converge to the same limit $\vec u$. This limit $\vec u$ belongs to the space X^3 where

$$\|f\|_X=\sup_{0<t}\|f(t,.)\|_{\dot M^{2,3}}+\sup_{0<t}\sqrt{t}\|f(t,.)\|_\infty \tag{46}$$

From (42) and (43), we see that

$$\|B(\vec f,\vec g)\|_X\le 2C_0\|\vec f\|_X\|\vec g\|_X \tag{47}$$

so that, if $\|e^{t\Delta}\vec u_0\|_X<\frac{1}{8C_0}$ there is a unique solution $\vec u$ in the ball $\|\vec u\|_X<\frac{1}{4C_0}$ of the fixed–point equation $\vec u=e^{t\Delta}\vec u_0-B(\vec u,\vec u)$. This solution will belong to $(\mathcal{C}((0,+\infty),L^2_{uloc}))^3$ for a general $\vec u\in(\dot M^{2,3})^3$ (small enough to ensure that $\|e^{t\Delta}\vec u_0\|_X<\frac{1}{8C_0}$); when $\vec u_0\in(E_2)^3$, then $\vec u$ belongs to $(\mathcal{C}([0,+\infty),E_2))^3$. Moreover, $\vec u_\epsilon$ is a solution of the fixed–point problem $\vec u_\epsilon=e^{t\Delta}\vec u_0-B(\omega_\epsilon*\vec u_\epsilon,\vec u_\epsilon)$. Since $\|f*\omega_\epsilon\|_X\le\|f\|_X$, we see that if $\|e^{t\Delta}\vec u_0\|_X\le\delta<\frac{1}{8C_0}$ then $\|\vec u\|_X\le 2\delta$ and $\|\vec u_\epsilon\|_X\le 2\delta$. We define $\vec w_\epsilon=\vec u_\epsilon-\vec u$. Then we have

$$\vec w_\epsilon=B(\vec u,\vec u)-B(\vec u_\epsilon*\omega_\epsilon,\vec u_\epsilon)=-B(\vec u_\epsilon*\omega_\epsilon,\vec w_\epsilon)-B(\vec w_\epsilon*\omega_\epsilon,\vec u)-B(\vec u*\omega_\epsilon-\vec u,\vec u) \tag{48}$$

We use (41) and get

$$\sup_{0<t<T}\|\vec w_\epsilon\|_{L^2_{uloc}}\le 4C_0\delta\sup_{0<t<T}\|\vec w_\epsilon\|_{L^2_{uloc}}+2C_0\delta\sup_{0<t<T}\|\vec u*\omega_\epsilon-\vec u\|_{L^2_{uloc}} \tag{49}$$

If $\vec u_0\in(E_2)^3$, we have $\vec u\in(\mathcal{C}([0,+\infty),E_2))^3$ hence $\lim_{\epsilon\to0}\sup_{0<t<T}\|\vec u*\omega_\epsilon-\vec u\|_{L^2_{uloc}}=0$.

Thus, Theorem 5 is proved. $\diamond$

Remark : we will see in the next section that if $f\in\dot M^{2,3}$ is homogeneous, then $f\in E_2$.

3. Large solutions.

When $\vec u_0$ is large, we know only a few things on the solutions of equations (1) :

Theorem 6 :

Let $\vec u_0\in(\dot M^{2,3}(\mathbb{R}^3))^3$ be such that $\vec\nabla.\vec u_0=0$. Then, there exists a constant C_0 (which doesn't depend on $\vec u_0$) such that the following assertions are true :
(A) [Existence] equations (1) have a suitable solution $\vec u$ on $(0,+\infty)\times\mathbb{R}^3$ (with pressure p given by $\vec\nabla p=-\sum_{i=1}^3\sum_{j=1}^3\vec\nabla\frac{1}{\Delta}\partial_i\partial_j(u_iu_j)$) such that

$$\sup_{x_0\in\mathbb{R}^3,\ R>0,\ t>0}\frac{1}{R+\sqrt{\frac{t}{T_0}}}\int_{|x-x_0|<R}|\vec u(t,x)|^2\,dx\le C_0\|\vec u_0\|^2_{\dot M^{2,3}} \tag{50}$$

and

$$\sup_{x_0\in\mathbb{R}^3,\ t>0}\sqrt{\frac{T_0}{t}}\int_0^t\int_{|x-x_0|<\sqrt{\frac{t}{T_0}}}|\vec\nabla\otimes\vec u(s,x)|^2\,dx\,ds\le C_0\|\vec u_0\|^2_{\dot M^{2,3}}. \tag{51}$$

where $T_0=\frac{1}{C_0^4\sup(1,\|\vec u_0\|^2_{\dot M^{2,3}})}$.
(B) [Local spatial boundedness] For every $T>0$ and every compact subset K of $\mathbb{R}^3$ we have $\vec u\in L^1_tL^\infty_x((0,T)\times K)$.
(C) [Boundedness] If moreover $\vec u_0\in(E_2)^3$ then for almost every positive t the function $\vec u(t,.)$ belongs to L^∞.
(D) [Homogeneous data] If $\vec u_0$ is homogeneous, then $\vec u_0\in(E_2)^3$ and solutions $\vec u$ descrtibed in point (A) satisfy

$$\sup_{t>0}\frac{1}{\sqrt t}\int_0^t\|\vec u(s,.)\|_\infty\,ds<+\infty \tag{52}$$

and

$$\textit{for some } R>0,\ \textit{and for every } (t,x) \textit{ such that } |x|>R\sqrt t,\quad |\vec u(t,x)|<\frac{1}{\sqrt t} \tag{53}$$

(E) [Self–similarity] If $\vec u$ is self-similar ($\lambda\vec u(\lambda^2 t,\lambda x)=\vec u(t,x)$), then the profile $\vec u(1,.)=\vec U$ is a bounded function.

Proof : Point (A) is given by theorem 4. Point (B) is a consequence of point (A) : if ω_K is a function in $\mathcal{D}(\mathbb{R}^3)$ which is equal to 1 on $\{x\in\mathbb{R}^3\ /\ d(x,K)\le 1\}$, we may estimate $\vec u$ on $(0,T)\times K$ by writing

$$\vec u=e^{t\Delta}\vec u_0-B(\omega_K\vec u,\vec u)-B((1-\omega_K)\vec u,\vec u) \tag{54}$$

where

$$|e^{t\Delta}\vec u_0|\le C\|\vec u_0\|_{\dot M^{2,3}}\frac{1}{\sqrt t} \tag{55}$$

and

$$|B((1-\omega_K)\vec u,\vec u)|\le C\int_0^t\int_{|x-y|\ge1}\frac{1}{|x-y|^4}|\vec u(s,y)|^2\ dy\ ds\le C't\sup_{0<s<t}\|\vec u(s,.)\|^2_{L^2_{uloc}} \tag{56}$$

In order to control $B(\omega_K\vec u,\vec u)$, we write K_1 for the support of ω_K; on $(0,T)\times K_1$, $\vec u$ belongs to $L^2_t\dot H^1_x$ so that $\vec u\otimes\vec u$ belongs to $L^1_t\dot B^{1/2,1}_2$ and thus $\frac{1}{\Delta}\mathbb{P}\vec\nabla.(\omega_k\vec u\otimes\vec u)$ belongs to $L^1_t\dot B^{3/2,1}_2$ on $(0,T)\times\mathbb{R}^3$; this gives (see [LEM 02]) that

$$\int_0^T\|B(\omega_K\vec u,\vec u)\|_{\dot B^{3/2,1}_2}\ dt\le C_K\int_0^T\int_{K_1}|\vec u|^2+|\vec\nabla\otimes\vec u|^2\ dx\ dt \tag{57}$$

and we have proved (B), since $\dot B^{3/2,1}_2\subset L^\infty$.

(C) is then a consequence of Theorems 3 and 1. (Same proof as inequality (34) in Theorem 4).

Let $f\in L^2_{uloc}$; then f is homogeneous (of homogeneity exponent -1) if and only if $f(x)=F(\frac{x}{|x|})\frac{1}{|x|}$ where $F\in L^2(S^2)$ (see [LEM 02]). Moreover, we have

$$\int_{|x-x_0|<1}|F(\frac{x}{|x|})|\frac{1}{|x|^2}\ dx\le C\int_{|\sigma-\frac{x_0}{|x_0|}|\le C\frac{1}{|x_0|}}|F(\sigma)|^2\ d\sigma \tag{58}$$

so that $f\in E_2$. Thus, we may apply point (C) and find that for some $R>0$ we have $|\vec u(t,x)|\le 1$ for $1/2<t<1$ and $|x|>R$. The value of R depends only on $\vec u_0$ and not on the specific solution $\vec u$. Thus, since $\lambda\vec u(\lambda^2t,\lambda x)$ is a solution as well which satisfies (50) and (51), we find that $|\vec u(t,x)|\le\lambda^{-1}$ for $t\in(\lambda^2/2,\lambda^2)$ and $|x|>R\lambda$. This gives (53). Then, we use (B) to get that $\vec u$ belongs to $L^1_tL^\infty_x$ on $(0,1)\times B(0,R)$ and (53) to get that $\vec u$ belongs to $L^1_tL^\infty_x$ on $(0,1)\times(\mathbb{R}^3-B(0,R))$. Thus, $\vec u\in L^1_tL^\infty_x$ on $(0,1)\times\mathbb{R}^3$ and moreover its norm is controlled by a constant which depend only on $\vec u_0$ and not on the specific solution $\vec u$. Thus, rescaling, we find that $\int_0^{\lambda^2}\|\vec u(t,.)\|_\infty\ dt\le C\lambda$. Thus (D) is proved.

(E) is a direct consequence of (D) : if $\vec u(t,x)=\frac{1}{\sqrt t}\vec U(\frac{x}{\sqrt t})$, then $\|\vec U\|_\infty=2\int_0^1\|\vec u(t,.)\|_\infty\ dt$. Thus, Theorem 6 is proved. ◇

Remark : Grujič [GRU 06] proved that the profile $\vec U$ of a self–similar suitable solution of equations (1) must be bounded on any compact subset of $\mathbb{R}^3$.

4. A scale-preserving approximation to the Navier-Stokes problem.

One of the main difficulty in the Navier–Stokes equations is the fact that p depends in a nonlocal way on $\vec u$. In formula (4), p is expressed through the use of singular integral operators whose kernels are supported by the whole space. In order to turn the equations in local equations, we will consider the following modification of the Navier–Stokes equations, associated to a positive ϵ :

$$\begin{cases}\partial_t\vec u_\epsilon+\vec\nabla.(\vec u_\epsilon\otimes\vec u_\epsilon)=\Delta\vec u_\epsilon-\vec\nabla p_\epsilon\\ \vec u_{\epsilon\ |t=0}=\vec u_0\\ p_\epsilon=-\frac{1}{\epsilon}\vec\nabla.\vec u_\epsilon\end{cases} \tag{59}$$

We will show that those equations are well fitted for small regular initial values and provide solutions which converge to the solution of equations (1). An important feature is that equations (59) are invariant under the same rescaling as equations (1). In particular, when $\vec u_0$ is homogeneous (and small), we shall have self–similar solutions for (59).

We shall work with an initial value $\vec{u}_0 \in (\dot{B}_\infty^{3/q-1,\infty})^3$ for some $q \in [1,3)$. Such Besov spaces have been studied by Cannone [CAN 95] as a good frame to exhibit self–similar solutions. Let us remark that the function $1/|x|$ belongs to $\dot{B}_\infty^{3/q-1,\infty}$ for every $q \geq 1$.

Theorem 7 :

Let $q \in [1,3)$. Then there exists a constant C_q such that, for every $\vec{u}_0 \in (\dot{B}_q^{3/q-1,\infty}(\mathbb{R}^3))^3$ such that, $\vec{\nabla}.\vec{u}_0 = 0$ and $\|\vec{u}_0\|_{\dot{B}_q^{3/q-1,\infty}} < C_q$, the following assertions are true :

(A) [Existence] equations (1) have a unique solution $\vec{u}$ on $(0,+\infty) \times \mathbb{R}^3$ such that

$$\sup_{t>0} \|\vec{u}(t,.)\|_{\dot{B}_q^{3/q-1,\infty}} \leq 2\|\vec{u}_0\|_{\dot{B}_q^{3/q-1,\infty}} \tag{60}$$

(with pressure p given by $\vec{\nabla}p = -\sum_{i=1}^3 \sum_{j=1}^3 \vec{\nabla}\frac{1}{\Delta}\partial_i\partial_j(u_iu_j)$)

(B) [Existence for the modified equations] For every $\epsilon > 0$, equations (59) have a unique solution $\vec{u}_\epsilon$ on $(0,+\infty)\times\mathbb{R}^3$ such that

$$\sup_{t>0} \|\vec{u}_\epsilon(t,.)\|_{\dot{B}_q^{3/q-1,\infty}} \leq 2\|\vec{u}_0\|_{\dot{B}_q^{3/q-1,\infty}} \tag{61}$$

(C) [Convergence] When ϵ goes to 0, the solutions $(\vec{u}_\epsilon, p_\epsilon)$ converge (in the sense of distributions) to the solution $(\vec{u}, p)$ of equations (1).

(D) [Self–similarity] If $\vec{u}_0$ is homogeneous, then $\vec{u}$ and $\vec{u}_\epsilon$ are self–similar.

Proof : We define L as the operator

$$Lf = \int_0^t e^{(t-s)\Delta}\Delta f(s,.)\ ds \tag{62}$$

and, for $\lambda > 0$, τ_λ as the operator

$$(\tau_\lambda f)(t,x) = f(\lambda t, x) \tag{63}$$

Thus $(\vec{u}, p)$ is a solution of (1) with $\vec{u} \in L_t^\infty \dot{B}_q^{3/q-1,\infty}$ and $p \in L_t^\infty \dot{B}_q^{3/q-2,\infty}$ if and only if

$$\begin{cases} \vec{u} = e^{t\Delta}\vec{u}_0 - L(\frac{1}{\Delta}\vec{\nabla}.(\vec{u}\otimes\vec{u})) - L(\frac{1}{\Delta}\vec{\nabla}p) \\ \\ p = -\frac{1}{\Delta}\vec{\nabla}\otimes\vec{\nabla}.(\vec{u}\otimes\vec{u}) \end{cases} \tag{64}$$

Similarly, taking the divergence of (59), we find that

$$\partial_t p_\epsilon = (1+\frac{1}{\epsilon})\Delta p_\epsilon + \frac{1}{\epsilon}\vec{\nabla}\otimes\vec{\nabla}.(\vec{u}_\epsilon\otimes\vec{u}_\epsilon) \tag{65}$$

so that (since $p_{\epsilon\ |t=0} = 0$) $(\vec{u}_\epsilon, p_\epsilon)$ is a solution of (59) if and only if

$$\begin{cases} \vec{u}_\epsilon = e^{t\Delta}\vec{u}_0 - L(\frac{1}{\Delta}\vec{\nabla}.(\vec{u}_\epsilon\otimes\vec{u}_\epsilon)) - L(\frac{1}{\Delta}\vec{\nabla}p_\epsilon) \\ \\ p_\epsilon = \int_0^t e^{(1+\frac{1}{\epsilon})(t-s)\Delta}\frac{1}{\epsilon}\vec{\nabla}\otimes\vec{\nabla}.(\vec{u}_\epsilon\otimes\vec{u}_\epsilon)\ ds = \frac{1}{1+\epsilon}\tau_{1+\frac{1}{\epsilon}}L(\frac{1}{\Delta}\vec{\nabla}\otimes\vec{\nabla}.\tau^{-1}_{1+\frac{1}{\epsilon}}\vec{u}\otimes\tau^{-1}_{1+\frac{1}{\epsilon}}\vec{u})) \end{cases} \tag{66}$$

It is now easy to conclude by using classical estimates [LEM 02] for $1 \leq q < 3$:

$$\|fg\|_{\dot{B}_q^{3/q-2,\infty}} \leq C_q\|f\|_{\dot{B}_q^{3/q-1,\infty}}\|g\|_{\dot{B}_q^{3/q-1,\infty}} \tag{67}$$

$$\|\frac{1}{\Delta}\partial_i f\|_{\dot{B}_q^{3/q-1,\infty}} \leq C_q\|f\|_{\dot{B}_q^{3/q-2,\infty}} \tag{68}$$

$$\|\frac{1}{\Delta}\partial_i\partial_j f\|_{\dot{B}_q^{3/q-2,\infty}} \leq C_q\|f\|_{\dot{B}_q^{3/q-2,\infty}} \tag{68}$$

(69) $$\|L(f)\|_{L^\infty_t \dot B_q^{3/q-1,\infty}} \le C_q \|f\|_{L^\infty_t \dot B_q^{3/q-1,\infty}}$$

(70) $$\|L(f)\|_{L^\infty_t \dot B_q^{3/q-2,\infty}} \le C_q \|f\|_{L^\infty_t \dot B_q^{3/q-2,\infty}}$$

(71) $$\|\tau_\lambda(f)\|_{L^\infty_t \dot B_q^{3/q-1,\infty}} = \|f\|_{L^\infty_t \dot B_q^{3/q-1,\infty}}$$

(72) $$\|\tau_\lambda(f)\|_{L^\infty_t \dot B_q^{3/q-2,\infty}} = \|f\|_{L^\infty_t \dot B_q^{3/q-2,\infty}}$$

Thus, we find that the bilinear operators

(73) $$A(\vec u, \vec v) = L(\frac{1}{\Delta}\vec\nabla.\vec u \otimes \vec v) - L(\frac{1}{\Delta}\vec\nabla(\frac{1}{\Delta}\vec\nabla \otimes \vec\nabla.(\vec u \otimes \vec v)))$$

and

(74) $$A_\epsilon(\vec u, \vec v) = L(\frac{1}{\Delta}\vec\nabla.\vec u \otimes \vec v) + \frac{1}{1+\epsilon} L(\frac{1}{\Delta}\vec\nabla(\tau_{1+\frac{1}{\epsilon}} L(\frac{1}{\Delta}\vec\nabla \otimes \vec\nabla.\tau^{-1}_{1+\frac{1}{\epsilon}}\vec u \otimes \tau^{-1}_{1+\frac{1}{\epsilon}}\vec u)))$$

are an equicontinuous family of bounded bilinear operators on $(L^\infty_t \dot B_q^{3/q-1,\infty})^3$. Thus, if $\vec u_0$ is small enough, we may find a solution to $\vec u = e^{t\Delta}\vec u_0 - A(\vec u, \vec u)$ or to $\vec u_\epsilon = e^{t\Delta}\vec u_0 - A_\epsilon(\vec u_\epsilon, \vec u_\epsilon)$. Thus, (A) and (B) (and (D)) are proved.

Since $\vec u_\epsilon$ remain bounded in $L^\infty_t \dot B_q^{3/q-1,\infty}$ and p_ϵ remain bounded in $L^\infty_t \dot B_q^{3/q-2,\infty}$, they belong to a relatively compact subset of $\mathcal D'((0,+\infty)\times \mathbb R^3)$. hence in order to prove convergence it is enough to check that if $(\vec v, q)$ is a limit of some sequence $(\vec u_{\epsilon_k}, p_{\epsilon_k})$ with $\epsilon_k \to 0$, then $\vec v = \vec u$ and $q = p$. It is even enough to show that $(\vec v, q)$ is a solution of (1). Since p_ϵ is bounded in $L^\infty_t \dot B_q^{3/q-2,\infty}$, we have that $\vec\nabla.\vec v = -\lim_{\epsilon_k\to 0}\epsilon_k p_{\epsilon_k} = 0$. Moreover, $\partial_t \vec v = \lim_{\epsilon_k\to 0}\partial_t \vec v_{\epsilon_k} = \Delta\vec v - \vec\nabla q - \lim_{\epsilon_k\to 0}\vec\nabla.(\vec v_{\epsilon_k}\otimes \vec v_{\epsilon_k})$. But $\vec v_\epsilon$ is bounded in $L^\infty_t \dot B_q^{3/q-1,\infty}$ and $\partial_t \vec v_\epsilon$ is bounded in $L^\infty_t \dot B_q^{3/q-3,\infty}$, hence there exists $\sigma < 0 < \tau$ such that, on any compact subset K of $(0,+\infty)\times\mathbb R^3$, $\vec v_\epsilon$ is bounded in $L^2_t H^\tau_x$ and $\partial_t \vec v_\epsilon$ is bounded in $L^2_t H^\sigma$, so that [LEM 02] $\vec v_\epsilon$ is bounded in $H^\rho_{t,x}(K)$ for some positive ρ; by Rellich's theorem, we get that $\vec v_{\epsilon_k}$ is strongly convergent in $L^2_{t,x}(K)$ for every compact K, hence $\lim_{\epsilon_k\to 0}\vec\nabla.(\vec v_{\epsilon_k}\otimes\vec v_{\epsilon_k}) = \vec\nabla.(\vec v\otimes\vec v)$. Hence, $\vec v$ is solution of $\partial_t\vec v = \Delta\vec v - \vec\nabla.(\vec v\otimes\vec v) - \vec\nabla q$ and $\vec\nabla.\vec v = 0$ and is small in $L^\infty_t \dot B_q^{3/q-1,\infty}$; but, then, $\vec v$ is smooth for $t>0$ and thus $\vec\nabla.(\vec v\otimes\vec v) = (\vec v.\vec\nabla)\vec v$. Theorem 7 is proved. ◇

5. Another scale-preserving approximation to the Navier-Stokes problem.

Equations (59) are not good when dealing with large data (and thus looking for weak solutions). Indeed, when we write the energy balance (8) the remainder R given by formula (9) is no longer equal to 0, since $\vec u$ is no longer divergence-free. The term $|\vec u|^2\vec\nabla.\vec u$ provides the worst contribution to the energy since it cannot be controlled by the (local) $L^\infty_t L^2_x$ and $L^2_t \dot H^1_x$ norms. We have to add a damping term to ensure that $|\vec u|^2$ belongs (locally) to $L^2_t L^2_x$. Vishik and Fursikov proposed the following approximation [VIS 77]

(75) $$\begin{cases} \partial_t \vec u_{\epsilon,\alpha} + (\vec u_{\epsilon,\alpha}.\vec\nabla)\vec u_{\epsilon,\alpha} = \Delta\vec u_{\epsilon,\alpha} - \alpha|\vec u_{\epsilon,\alpha}|^4\vec u_{\epsilon,\alpha} - \vec\nabla p_{\epsilon,\alpha} \\ \vec u_{\epsilon,\alpha\ |t=0} = \vec u_0 \\ p_{\epsilon,\alpha} = -\frac{1}{\epsilon}\vec\nabla.\vec u_{\epsilon,\alpha} \end{cases}$$

where $\epsilon > 0$ and $\alpha > 0$.

This model,which provides unique solutions for a large class of initial values, is not well fitted to homogeneous initial values,since the equations are not invariant through the rescaling. Thus, we shall study a modified model :

(76) $$\begin{cases} \partial_t \vec u_{\epsilon,\alpha} + (\vec u_{\epsilon,\alpha}.\vec\nabla)\vec u_{\epsilon,\alpha} = \Delta\vec u_{\epsilon,\alpha} - \alpha|\vec u_{\epsilon,\alpha}|^2\vec u_{\epsilon,\alpha} - \vec\nabla p_{\epsilon,\alpha} \\ \vec u_{\epsilon,\alpha\ |t=0} = \vec u_0 \\ p_{\epsilon,\alpha} = -\frac{1}{\epsilon}\vec\nabla.\vec u_{\epsilon,\alpha} \end{cases}$$

under some additional restriction on α and ϵ (namely, $\epsilon < 4\alpha$). We shall loose uniqueness, but keep scaling invariance (as for (1)) and get energy equality (in contrast to (1)). Moreover, we shall prove convergence to suitable solutions of (1) when ϵ goes to 0 (and when $\vec{u}_0 \in (\dot{M}^{2,3})^3$).

Equations (76) have been studied by F. Lelièvre [LEL 10] in the case of finite-energy ($\vec{u}_0 \in (L^2)^3$) and in the case of uniformly square-integrable initial value ($\vec{u}_0 \in (L^2_{uloc})^3$). He proved the existence of global weak solutions in the first case and of local weak solutions in the second case (with existence time depending on α). Those solutions are locally $L^4_t L^4_x$ and we can write the energy balance

$$\partial_t |\vec{u}_{\epsilon,\alpha}|^2 + 2|\vec{\nabla} \otimes \vec{u}_{\epsilon,\alpha}|^2 = \sum_{i=1}^{3} \partial_i (2\vec{u}_{\epsilon,\alpha}.\partial_i \vec{u}_{\epsilon,\alpha} - (|\vec{u}_{\epsilon,\alpha}|^2 + 2p_{\epsilon,\alpha}) u_{\epsilon,\alpha,i}) + R \tag{77}$$

where

$$R = -2\alpha|\vec{u}_{\epsilon,\alpha}|^4 + |\vec{u}_{\epsilon,\alpha}|^2 \vec{\nabla}.\vec{u}_{\epsilon,\alpha} - \frac{2}{\epsilon}|\vec{\nabla}.\vec{u}_{\epsilon,\alpha}|^2 \le -\alpha|\vec{u}_{\epsilon,\alpha}|^4 + (\frac{1}{4\alpha} - \frac{2}{\epsilon})|\vec{\nabla}.\vec{u}_{\epsilon,\alpha}|^2 \le -\alpha|\vec{u}_{\epsilon,\alpha}|^4 - \frac{1}{\epsilon}|\vec{\nabla}.\vec{u}_{\epsilon,\alpha}|^2 \le 0. \tag{78}$$

This is the key tool to exhibit weak solutions. Let us recall the main results of [LEL 10] :

Theorem 8 :
Let $0 < \epsilon < 4\alpha$. Let $\vec{u}_0 \in (L^2(\mathbb{R}^3))^3$ be such that $\vec{\nabla}.\vec{u}_0 = 0$. Then the following assertions are true :
(A) [Existence] Equations (76) have a solution $\vec{u}_{\epsilon,\alpha}$ on $(0, +\infty) \times \mathbb{R}^3$ such that $\vec{u}_{\epsilon,\alpha} \in (L^\infty_t L^2_x)^3 \cap (L^2_t \dot{H}^1_x)^3 \cap (L^4_t L^4_x)^3$.
(B) [Energy inequality] For every $t > 0$, we have

$$\|\vec{u}_{\epsilon,\alpha}(t,.)\|_2^2 + 2\int_0^t \|\vec{\nabla} \otimes \vec{u}_{\epsilon,\alpha}\|_2^2 \, ds + \alpha \int_0^t \|\vec{u}_{\epsilon,\alpha}\|_4^4 \, ds + \frac{1}{\epsilon}\int_0^t \|\vec{\nabla}.\vec{u}_{\epsilon,\alpha}\|_2^2 \, ds \le \|\vec{u}_0\|_2^2 \tag{79}$$

(C) [Convergence] There exists a sequence (α_k, ϵ_k) going to $(0,0)$ such that $(\vec{u}_{\epsilon_k,\alpha_k}, p_{\epsilon_k,\alpha_k})$ converges (in the sense of distributions) to a suitable solution of (1).

Theorem 9 :
Let $0 < \epsilon < 4\alpha$. Let $\vec{u}_0 \in (L^2_{uloc}(\mathbb{R}^3))^3$ be such that $\vec{\nabla}.\vec{u}_0 = 0$. Then, the following assertions are true :
(A) [Existence] Equations (76) have a solution $\vec{u}_{\epsilon,\alpha}$ on $(0, T_\epsilon) \times \mathbb{R}^3$ such that $\vec{u}_{\epsilon,\alpha} \in ((L^\infty_t L^2_x)_{uloc})^3 \cap ((L^2_t \dot{H}^1_x)_{uloc})^3 \cap ((L^4_t L^4_x)_{uloc})^3$.
(B) [Uniqueness] If $\alpha > 6$ and $\epsilon < \alpha/6$, then if $\vec{u}$ and $\vec{v}$ are two solutions of (76) on $(0,T) \times \mathbb{R}^3$ which belong to $((L^\infty_t L^2_x)_{uloc})^3 \cap ((L^2_t \dot{H}^1_x)_{uloc})^3 \cap ((L^4_t L^4_x)_{uloc})^3$, then $\vec{u} = \vec{v}$.
(C) [Self–similarity] If $\alpha > 6$ and $\epsilon < \alpha/6$ and if $\vec{u}_0$ is homogeneous, then $\vec{u}_{\epsilon,\alpha}$ is self–similar.

The problem in Theorem 9 is that we have no control when (α, ϵ) goes to $(0,0)$, neither on the existence time T_ϵ nor on the size of the solution. Indeed, we have to control the pressure $p_{\epsilon,\alpha}$; in the case of finite energy solutions, this can be done through a result on maximal regularity of the heat kernel; this results breaks down in the case of uniformly locally square-integrable solutions. But we can recover it in the case of an initial value in the Morrey–Campanato space :

Theorem 10 :
Let $0 < \epsilon < 4\alpha$. Let $\vec{u}_0 \in (\dot{M}^{2;3}(\mathbb{R}^3))^3$ be such that $\vec{\nabla}.\vec{u}_0 = 0$. Then, the following assertions are true :
(A) [Existence] Equations (76) have a solution $\vec{u}_{\epsilon,\alpha}$ on $(0, +\infty) \times \mathbb{R}^3$ such that for every $T > 0$ we have $\vec{u}_{\epsilon,\alpha} \in ((L^\infty_t L^2_x)_{uloc})^3 \cap ((L^2_t \dot{H}^1_x)_{uloc})^3 \cap ((L^4_t L^4_x)_{uloc})^3$ on $(0,T) \times \mathbb{R}^3$.
(B) [Convergence] There exists a sequence (α_k, ϵ_k) going to $(0,0)$ such that $(\vec{u}_{\epsilon_k,\alpha_k}, p_{\epsilon_k,\alpha_k})$ converges (in the sense of distributions) to a suitable solution of (1).

Proof :
From Theorem 9, we may derive by rescaling, compactness and extraction that the equations (76) with initial data in $\dot{M}^{2,3}$ have a global solution $\vec{u}_{\epsilon,\alpha}$ whose size depends on ϵ and α as well as on $\|\vec{u}_0\|_{\dot{M}^{2,3}}$. As this diagonal extraction must be done very carefully (since it interacts with multiple rescalings), we describe it in the next section. Let us take this existence for granted. We shall try to get rid of the dependence on ϵ and α.

The problem is to find a space X such that $\dot{M}^{2,3} \subset X$ and such that the solution $\vec{u}_{\epsilon,\alpha}$ of equations (76) will be well controlled through the estimates on X norms on some strip $(0,T) \times \mathbb{R}^3$ for some time T independent from ϵ and α (with a (local in t and x) control in $L^\infty_t L^2 \cap L^2 \dot{H}^1_x$ independent from ϵ and α).

The space X we shall consider is the space $X = L^2(\omega(x)\ dx)$, where $\omega(x) = (1+|x|^2)^{-\lambda/2}$ with $\lambda \in (1,2)$. Since $\lambda > 1$, it is easy to check that $\int |\vec u_0|^2\ \omega(x)\ dx < C\|\vec u_0\|^2_{\dot M^{2,3}}$.

We shall use another weight : $\varpi(x) = (1+|x|^2)^{-\mu/2}$ with $\mu \in (3\frac{\lambda}{2}, \frac{3}{2} + 3\frac{\lambda}{4}) \subset (3/2, 3)$.

We start from the identity

$$\partial_t(|\vec u_{\epsilon,\alpha}|^2\omega) = 2\omega\vec u_{\epsilon,\alpha}.\Delta\vec u_{\epsilon,\alpha} - 2\omega\vec u_{\epsilon,\alpha}.(\vec u_{\epsilon,\alpha}.\vec\nabla\vec u_{\epsilon,\alpha}) - 2\alpha|\vec u_{\epsilon,\alpha}|^4\omega + \omega\frac{2}{\epsilon}\vec u_{\epsilon,\alpha}.\vec\nabla(\vec\nabla.\vec u_{\epsilon,\alpha}) \tag{80}$$

with

$$2\omega\vec u_{\epsilon,\alpha}.\Delta\vec u_{\epsilon,\alpha} = \Delta(\omega|\vec u_{\epsilon,\alpha}|^2) - |\vec u_{\epsilon,\alpha}|^2\Delta\omega - 2|\vec\nabla\otimes\vec u_{\epsilon,\alpha}|^2\omega - 2\sum_{i=1}^3 \partial_i\big(|\vec u_{\epsilon,\alpha}|^2\partial_i\omega\big) \tag{81}$$

$$2\omega\vec u_{\epsilon,\alpha}.(\vec u_{\epsilon,\alpha}.\vec\nabla\vec u_{\epsilon,\alpha}) = \vec\nabla\big(\omega|\vec u_{\epsilon,\alpha}|^2\vec u_{\epsilon,\alpha}\big) - \omega|\vec u_{\epsilon,\alpha}|^2\ \vec\nabla.\vec u_{\epsilon,\alpha} - |\vec u|^2\sum_{i=1}^3 u_{\epsilon,\alpha,i}\partial_i\ \omega \tag{82}$$

and

$$\omega\frac{2}{\epsilon}\vec u_{\epsilon,\alpha}.\vec\nabla\vec\nabla.\vec u_{\epsilon,\alpha} = \vec\nabla.\big(\omega\frac{2}{\epsilon}\vec\nabla.\vec u_{\epsilon,\alpha})\ \vec u_{\epsilon,\alpha}\big) - \frac{2}{\epsilon}|\vec\nabla.\vec u_{\epsilon,\alpha}|^2\omega - \frac{2}{\epsilon}\vec\nabla.\vec u_{\epsilon,\alpha}\sum_{i=1}^3 u_{\epsilon,\alpha,i}\partial_i\omega \tag{83}$$

We then introduce

$$U(t) = \int |\vec u_{\epsilon,\alpha}(t,x)|^2\omega(x)\ dx \tag{84}$$

$$V(t) = \int_0^t\int |\vec\nabla\otimes\vec u_{\epsilon,\alpha}(s,x)|^2\omega(x)\ ds\ dx \tag{85}$$

$$W(t) = \alpha\int_0^t\int |\vec u_{\epsilon,\alpha}(s,x)|^4\omega(x)\ ds\ dx \tag{86}$$

$$X(t) = \int_0^t\int \frac{1}{\epsilon}|\vec\nabla.\vec u_{\epsilon,\alpha}(s,x)|^2\ \omega(x)\ ds\ dx \tag{87}$$

$$Y(t) = \int_0^t\int \frac{1}{\epsilon^{3/2}}|\vec\nabla.\vec u_{\epsilon,\alpha}(s,x)|^{3/2}\varpi(x)\ ds\ dx \tag{88}$$

Integrating (80) in t and x (and writing $|\vec u_{\epsilon,\alpha}|^2|\vec\nabla.\vec u_{\epsilon,\alpha}| \le \alpha|\vec u_{\epsilon,\alpha}|^4 + \frac{1}{4\alpha}|\vec\nabla.\vec u_{\epsilon,\alpha}|^2 \le \alpha|\vec u_{\epsilon,\alpha}|^4 + \frac{1}{\epsilon}|\vec\nabla.\vec u_{\epsilon,\alpha}|^2$), we get

$$U(t) + 2V(t) + W(t) + X(t) \le U(0) + Z(t) \tag{89}$$

with

$$Z = \int_0^t\int |\vec u_{\epsilon,\alpha}|^2\Delta\omega\ ds\ dx + \int_0^t\int |\vec u_{\epsilon,\alpha}|^2\sum_{i=1}^3 u_i\partial_i\ \omega\ ds\ dx - \int_0^t\int \frac{2}{\epsilon}\vec\nabla.\vec u_{\epsilon,\alpha}\sum_{i=1}^3 u_{\epsilon,\alpha,i}\partial_i\omega\ ds\ dx = Z_1 + Z_2 + Z_3 \tag{90}$$

We have $|\Delta\omega(x)| \le C\omega(x)$, so that :

$$Z_1 \le C\int_0^t U(s)\ ds. \tag{91}$$

Since $\lambda < 2$, we have $|\vec\nabla\omega(x)| \leq \lambda(1+|x|^2)^{-1/2}\omega(x) \leq \lambda\omega(x)^{3/2}$so that :

$$Z_2 \leq C\int_0^t\int |\vec u_{\epsilon,\alpha}(s,x)|^3\omega^{3/2}\ ds\ dx \tag{92}$$

hence

$$Z_2 \leq C'\int_0^t U(s)^{3/4}\|\sqrt{\omega}\vec u_{\epsilon,\alpha}\|_{H^1}^{3/2}\ ds \leq \frac{1}{4}\int_0^t \|\sqrt{\omega}\vec u_{\epsilon,\alpha}\|_{H^1}^2\ ds + C''\int_0^t U(s)^3\ ds \tag{93}$$

and finally

$$Z_2 \leq \frac{1}{4}V(t) + C\int_0^t U(s)\ ds + C\int_0^t U^3(s)\ ds. \tag{94}$$

Since $2\mu/3 \leq 1+\lambda/2$, we have $|\vec\nabla\omega(x)| \leq \lambda(1+|x|^2)^{-1/2}\omega(x) \leq \lambda\omega^{1/2}\varpi^{2/3}$ so that

$$Z_3 \leq C\int_0^t \|\sqrt{\omega}\vec u_{\epsilon,\alpha}\|_3\|\frac{1}{\epsilon}\vec\nabla.\vec u_{\epsilon,\alpha}\|_{3/2}\ ds \leq \frac{1}{4}V(t) + C\int_0^t U(s)\ ds + C\int_0^t U^3(s)\ ds + CY(t) \tag{95}$$

The main problem we have to deal with is to control the size of $Y(t)$ indepedently of ϵ and α. We define $p_{\epsilon,\alpha} = -\frac{1}{\epsilon}\vec\nabla.\vec u_{\epsilon,\alpha}$ (so that $Y(t) = \int_0^t\int |p_{\epsilon,\alpha}(s,x)|^{3/2}\varpi(x)\ ds\ dx$) and we write

$$p_{\epsilon,\alpha} = -\frac{1}{1+\epsilon}\int_0^t e^{(1+\frac{1}{\epsilon})(t-s)\Delta}(1+\frac{1}{\epsilon})\Delta\frac{\vec\nabla}{\Delta}.(\vec u_{\epsilon,\alpha}.\vec\nabla\vec u_{\epsilon,\alpha} + \alpha|\vec u_{\epsilon,\alpha}|^2\vec u_{\epsilon,\alpha})\ ds \tag{96}$$

Since $0<\mu<3$, the weight ϖ belongs to the Muckenhoupt class $A_{3/2}$ [STE 93]; thus, we have maximal regularity for the heat kernel on $L^{3/2}_t L^{3/2}_x(\varpi(x)\ dx)$: the operator T defined by

$$f \mapsto T(f) = \int_0^t e^{(t-s)\Delta}\Delta f(s,.)\ ds \tag{97}$$

is a Calderón–Zygmund operator on the homogeneous–type space $\mathbb{R}\times\mathbb{R}^3$ (endowed with the Lebesgue measure $dt\ dx$ and the pseudo–distance $\delta((t,x),(t',x')) = \big(|t-t'|^2+|x-x'|^4\big)^{1/4}$) [LEM 02]; since $\varpi(x)$ is a Muckenhoupt weight on $\mathbb{R}\times\mathbb{R}^3$, the operator T is bounded on $L^{3/2}(dt\ dx)$[STE 93] [PRA 07]. We thus get

$$Y(t) \leq C\int_0^t\int |\frac{\vec\nabla}{\Delta}.(\vec u_{\epsilon,\alpha}.\vec\nabla\vec u_{\epsilon,\alpha})\big|^{3/2}\ \varpi(x)\ ds\ dx + C\int_0^t\int |\frac{\vec\nabla}{\Delta}.(\alpha|\vec u_{\epsilon,\alpha}|^2\vec u_{\epsilon,\alpha})\big|^{3/2}\ \varpi(x)\ ds\ dx \tag{98}$$

We write $f_1 = |\vec\nabla\otimes\vec u_{\epsilon,\alpha}|$, $f_2 = \sqrt{\alpha}|\vec u_{\epsilon,\alpha}|^2$ and $g = |\vec u_{\epsilon,\alpha}|$ and thus we have

$$Y(t) \leq C\int_0^t\int |\frac{1}{\sqrt{-\Delta}}(f_1 g)\big|^{3/2}\ \varpi(x)\ ds\ dx + C\sqrt{\eta}\int_0^t\int |\frac{1}{\sqrt{-\Delta}}((f_2 g)\big|^{3/2}\ \varpi(x)\ ds\ dx \tag{99}$$

We define $E_j = B(0,2^j)$, $F_j = B(0,3\ 2^j)$ and $G_j = \mathbb{R}^3 - F_j$. We write

$$|\varpi^{2/3}\frac{1}{\sqrt{-\Delta}}(f_i g)| \leq C\sum_{j=0}^{+\infty} 2^{-2\mu j/3}1_{E_j}|\frac{1}{\sqrt{-\Delta}}(f_i g)| \tag{100}$$

with

$$2^{\frac{-2\mu j}{3}}1_{E_j}|\frac{1}{\sqrt{-\Delta}}(f_i g)| \leq 2^{\frac{-2\mu j}{3}}1_{E_j}|\frac{1}{\sqrt{-\Delta}}(1_{F_j}f_i g)| + 2^{\frac{-2\mu j}{3}}1_{E_j}|\frac{1}{\sqrt{-\Delta}}(1_{G_j}f_i g)| = A_j + B_j \tag{101}$$

In order to estimate A_j, we write

$$A_j \leq C2^{j(\lambda-2\frac{\mu}{3})}1_{B(0,2^j)}\frac{1}{\sqrt{-\Delta}}(\omega f_i g) \tag{102}$$

Since $\sqrt{\omega}g$ belongs to $L^2\cap L^6$,we find that, for $q\in(2,6)$, $\sqrt{\omega}g\in L^q$ so that $\omega f_i g\in L^r$ with $1/r=1/q+1/2$, which gives $\frac{1}{\sqrt{-\Delta}}(\omega f_i g)\in L^\rho$ with $1/\rho=1/q+1/6$ and finally

$$\|A_j\|_{L^{3/2}(x)}\leq C2^{j(\lambda-2\frac{\mu}{3}+\frac{3}{2}-\frac{3}{q})}\|\sqrt{\omega}f_i\|_2\|\sqrt{\omega}g\|_q. \tag{103}$$

Let $\theta=\frac{2\mu}{3}-\lambda-\frac{3}{2}+\frac{3}{q}$. As $\lambda<2\frac{\mu}{3}$, we may choose q close enough to 2 to ensure that $\theta>0$. We then have

$$\int_0^t\int|\sum_{j=0}^{+\infty}A_j|^{3/2}\,ds\,dx\leq C(\sum_{j=0}^{+\infty}2^{-j\theta})^{3/2}\int_0^t\|\sqrt{\omega}f_i\|_2^{3/2}\|\sqrt{\omega}g\|_q^{3/2}\,ds \tag{104}$$

If $2<q<18/7$, we may find an exponent $R>1$ such that, for all $\kappa>0$,

$$\|\sqrt{\omega}f_i\|_2^{3/2}\|\sqrt{\omega}g\|_q^{3/2}\leq\kappa\|\sqrt{\omega}f_i\|_2^2+\kappa\|\sqrt{\omega}g\|_6^2+C_{\kappa,q}\|\sqrt{\omega}g\|_2^{2R} \tag{105}$$

In order to estimate B_j, we write that

$$B_j\leq C2^{\frac{-2j\mu}{3}}1_{B(0,2^j)}\sum_{k=0}^{+\infty}\int_{3\ 2^{j+k}<|y|<3\ 2^{j+k+1}}2^{-(j+k)2}2^{(j+k)\lambda}(\omega f_i g)\,dy \tag{106}$$

and, since $\lambda<2$, we get that

$$B_j\leq C2^{\frac{-2j\mu}{3}}1_{B(0,2^j)}\|\sqrt{\omega}f_i\|_2\|\sqrt{\omega}g\|_2\sum_{k=0}^{+\infty}2^{-(j+k)2}2^{(j+k)\lambda}\leq C'2^{j(\lambda-2-\frac{2\mu}{3})}1_{B(0,2^j)}\|\sqrt{\omega}f_i\|_2\|\sqrt{\omega}g\|_2 \tag{107}$$

Since $\|B_j\|_{L^{3/2}(dx)}\leq C2^{2j}\|B_j\|_\infty$ and $\lambda<2\mu/3$,, we get that

$$\int_0^t\int|\sum_{j=0}^{+\infty}B_j|^{3/2}\,ds\,dx\leq C(\sum_{j=0}^{+\infty}2^{j(\lambda-\frac{2\mu}{3})})^{3/2}\int_0^t\|\sqrt{\omega}f_i\|_2^{3/2}\|\sqrt{\omega}g\|_2^{3/2}\,ds \tag{108}$$

where, for all $\kappa>0$,

$$\|\sqrt{\omega}f_i\|_2^{3/2}\|\sqrt{\omega}g\|_2^{3/2}\leq\frac{3}{4}\kappa\|\sqrt{\omega}f_i\|_2^2+\frac{1}{4\kappa^3}\|\sqrt{\omega}g\|_2^6 \tag{109}$$

Finally, we get

$$Z(t)\leq V(t)+\frac{1}{2}W(t)+C\int_0^t U(s)+U^3(s)+U^R(s)\,ds \tag{110}$$

so that

$$U(t)+V(t)+\frac{1}{2}W(t)+X(t)\leq U(0)+C\int_0^t U(s)+U^3(s)+U^R(s)\,ds \tag{111}$$

and we may conclude.

Thus far, we have got size estimates of the solution $\vec{u}_{\epsilon,\alpha}$ independent from ϵ and α for (t,x) in a strip $(0,T_0)\times\mathbb{R}^3$ where T_0 depends on the size of $\|\vec{u}_0\|_{\dot{M}^{2,3}}$. But we may rescale $\vec{u}_0$ in $\vec{u}_0(x)=\gamma\vec{u}_{0,\gamma}(\gamma x)$ with $\|\vec{u}_{0,\gamma}\|_{\dot{M}^{2,3}}=\|\vec{u}_0\|_{\dot{M}^{2,3}}$; this gives estimates on the strip $0<t<\gamma^{-2}T_0$. Let us notice that the weights ω and $\omega(x/\gamma)$ give way to the same Lebesgue spaces so that we have controls in $L^2(\omega dx)$ norm on $(0,\gamma^{-2}T_0)$ for every $\gamma>0$.

Thus, there exists a sequence (α_k,ϵ_k) going to $(0,0)$ such that $(\vec{u}_{\epsilon_k,\alpha_k})$ converges (in the sense of distributions) to some function $\vec{u}$ while $(p_{\epsilon_k,\alpha_k})$ converges weakly to some function p. Since moreover $\alpha^{1/4}\vec{u}_{\epsilon,\alpha}$ is bounded in every $L^4((0,T),L^4(\omega\,dx))$ and $\epsilon^{-1/2}\vec{\nabla}.\vec{u}_{\epsilon,\alpha}$ is bounded in every $L^2((0,T),L^2(\omega\,dx))$, we find that all terms but one in equation (76) have a limit in $\mathcal{D}'$ (and thus the last one as well) and we get the equation

$$\begin{cases}\partial_t\vec{u}+\lim\,(\vec{u}_{\epsilon_k,\alpha_k}.\vec{\nabla})\vec{u}_{\epsilon_k,\alpha_k}=\Delta\vec{u}-\vec{\nabla}p\\ \vec{u}_{|t=0}=\vec{u}_0\\ \vec{\nabla}.\vec{u}=0\end{cases} \tag{112}$$

Of course, we want to show that

$$\lim\, (\vec{u}_{\epsilon_k,\alpha_k}.\vec{\nabla})\vec{u}_{\epsilon_k,\alpha_k} = (\vec{u}.\vec{\nabla})\vec{u}. \tag{113}$$

Thus, we need to prove that $\vec{u}_{\epsilon_k,\alpha_k}$ converges to $\vec{u}$ strongly locally in $L^2_{t,x}$. We split $\vec{u}_{\epsilon_k,\alpha_k}$ in $\mathbb{P}\vec{u}_{\epsilon_k,\alpha_k}+(Id-\mathbb{P})\vec{u}_{\epsilon_k,\alpha_k}$ where $\mathbb{P}$ is the Leray projection operator. The operator $\mathbb{P}$ is bounded on $L^2(\omega\ dx)$, since ω belongs to the Muckenhoupt class $\mathcal{A}_2$. On every compact subset of $[0,+\infty)\times\mathbb{R}^3$, we control (uniformly in ϵ and α)) the size of $\mathbb{P}\vec{u}_{\epsilon,\alpha}$ in $L^2_t H^1_x$ and the size of $\partial_t\mathbb{P}\vec{u}_{\epsilon,\alpha}$ in $L^2_t H^{-2}_x$. A classical Rellich compactness argument [LEM 02] then ensures that $\mathbb{P}(\vec{u}_{\epsilon_k,\alpha_k})$ converges strongly locally in $L^2_{t,x}$ norm. We now consider the remaining part $\vec{v}_{\epsilon_k,\alpha_k} = (Id-\mathbb{P})\vec{u}_{\epsilon_k,\alpha_k}$. We have

$$\vec{v}_{\epsilon_k,\alpha_k} = \frac{\vec{\nabla}}{\Delta}\vec{\nabla}.\vec{u}_{\epsilon_k,\alpha_k}. \tag{114}$$

We want to show that, for every $x_0\in\mathbb{R}^3$ and every $T>0$, $\vec{v}_{\epsilon_k,\alpha_k}$ converges to 0 in $L^2((0,T)\times B(x_0,1))$. We consider a function $\theta\in\mathcal{D}(\mathbb{R}^3)$ such that $\theta(x)=1$ on $B(0,1)$ and $\theta(x)=0$ for $|x|>2$. We write $\theta_k(x)=\theta(\frac{x-x_0}{R_k})$ for some $R_k>10$ which will be fixed later. We define $\omega_0(x)=(1+|x|^2)^{-\lambda_0/2}$ with $\lambda\in(\lambda,2)$. We then write $\vec{v}_{\epsilon_k,\alpha_k} = A_k+B_k+C_k$ with

$$A_k = \frac{\vec{\nabla}}{\Delta}\big(\theta_k\vec{\nabla}.\vec{u}_{\epsilon_k,\alpha_k}\big),\quad B_k=\frac{\vec{\nabla}}{\Delta}\big(\vec{u}_{\epsilon_k,\alpha_k}.\vec{\nabla}\theta_k\big),\quad C_k=\frac{\vec{\nabla}}{\Delta}\vec{\nabla}.\big((1-\theta_k)\vec{u}_{\epsilon_k,\alpha_k}\big) \tag{115}$$

We then write

$$\|A_k\|_{L^2((0,T)\times B(x_0,1))} \le C\|A_k\|_{L^2_tL^6_x((0,T)\times\mathbb{R}^3)} \le C'\|\theta_k\vec{\nabla}.\vec{u}_{\epsilon_k,\alpha_k}\|_{L^2((0,T)\times\mathbb{R}^3)} \tag{116}$$

which gives

$$A_k \le C(R_k+|x_0|)^{\lambda/2}\epsilon_k^{1/2}\epsilon_k^{-1/2}\|\vec{\nabla}.\vec{u}_{\epsilon_k,\alpha_k}\|_{L^2((0,T),L^2(\omega\ dx))} \le C_T(R_k+|x_0|)^{\lambda/2}\epsilon_k^{1/2} \tag{117}$$

where C_T depends only on T and on $\|\vec{u}_0\|_{\dot{M}^{2,3}}$. We write in a similar way

$$\|B_k\|_{L^2((0,T)\times B(x_0,1))} \le C\|B_k\|_{L^2_tL^6_x((0,T)\times\mathbb{R}^3)} \le C'\|\vec{u}_{\epsilon_k,\alpha_k}.\vec{\nabla}\theta_k\|_{L^2((0,T)\times\mathbb{R}^3)} \tag{118}$$

which gives

$$B_k \le C(R_k+|x_0|)^{\lambda/2}R_k^{-1}\|\vec{u}_{\epsilon_k,\alpha_k}\|_{L^2((0,T),L^2(\omega\ dx))} \le C_T(R_k+|x_0|)^{\lambda/2}R_k^{-1} \tag{119}$$

Finally, we use the fact that ω_0 belongs to the Muckenhoupt class $\mathcal{A}_2$ and we write

$$\|C_k\|_{L^2((0,T)\times B(x_0,1))} \le C(1+|x_0|)^{\lambda_0/2}\|C_k\|_{L^2(0,T),L^2(\omega_0\ dx)} \le C'(1+|x_0|)^{\lambda_0/2}\|(1-\theta_k)\vec{u}_{\epsilon_k,\alpha_k}\|_{L^2(0,T),L^2(\omega_0\ dx)} \tag{120}$$

which gives

$$C_k \le C(1+|x_0|)^{\lambda_0/2}(R_k+|x_0|)^{(\lambda-\lambda_0)/2}\|\vec{u}_{\epsilon_k,\alpha_k}\|_{L^2((0,T),L^2(\omega\ dx))} \le C_T(1+|x_0|)^{\lambda_0/2}(R_k+|x_0|)^{(\lambda-\lambda_0)/2} \tag{121}$$

We take $R_k=\epsilon_k^{-1/2}$ and we find that

$$\|\vec{v}_{\epsilon_k,\alpha_k}\|_{L^2((0,T)\times B(x_0,1))} = O(\epsilon_k^{\frac{2-\lambda}{4}}) + O(\epsilon_k^{\frac{\lambda_0-\lambda}{4}}) \tag{122}$$

so that we have $\lim_{k\to+\infty}\|\vec{v}_{\epsilon_k,\alpha_k}\|_{L^2((0,T)\times B(x_0,1))}=0$.

It remains to show that the solution $\vec{u}$ we have obtained is a suitable one. Starting from the equation

$$\partial_t(|\vec{u}_{\epsilon,\alpha}|^2) = \Delta(|\vec{u}_{\epsilon,\alpha}|^2) - 2|\vec{\nabla}\otimes\vec{u}_{\epsilon,\alpha}|^2 - 2\vec{u}_{\epsilon,\alpha}.(\vec{u}_{\epsilon,\alpha}.\vec{\nabla}\vec{u}_{\epsilon,\alpha}) - 2\alpha|\vec{u}_{\epsilon,\alpha}|^4 - 2\vec{u}_{\epsilon,\alpha}.\vec{\nabla}p_{\epsilon,\alpha} \tag{123}$$

rewritten as

$$\partial_t(|\vec{u}_{\epsilon,\alpha}|^2) = \Delta(|\vec{u}_{\epsilon,\alpha}|^2) - 2|\vec{\nabla}\otimes\vec{u}_{\epsilon,\alpha}|^2 - \vec{\nabla}.((|\vec{u}_{\epsilon,\alpha}|^2+2p_{\epsilon,\alpha})\vec{u}_{\epsilon,\alpha}) + |\vec{u}_{\epsilon,\alpha}|^2\vec{\nabla}.\vec{u}_{\epsilon,\alpha} - 2\alpha|\vec{u}_{\epsilon,\alpha}|^4 + 2p_{\epsilon,\alpha}\vec{\nabla}.\vec{u}_{\epsilon,\alpha} \tag{124}$$

and noticing that

$$|\vec{u}_{\epsilon,\alpha}|^2\vec{\nabla}.\vec{u}_{\epsilon,\alpha} - 2\alpha|\vec{u}_{\epsilon,\alpha}|^4 + 2p_{\epsilon,\alpha}\vec{\nabla}.\vec{u}_{\epsilon,\alpha} \leq -\epsilon/2\ (|\vec{u}_{\epsilon,\alpha}|^4 + 4p_{\epsilon,\alpha}^2 + 2p_{\epsilon,\alpha}|\vec{u}_{\epsilon,\alpha}|^2) \leq 0 \tag{125}$$

we find (for some subsequence $(\epsilon_k;\alpha_k)$) that $\partial_t(|\vec{u}_{\epsilon,\alpha}|^2)$ converges to $\partial_t|\vec{u}|^2$, that $\Delta(|\vec{u}_{\epsilon,\alpha}|^2)$ converges to $\Delta|\vec{u}|^2$, that $|\vec{\nabla}\otimes\vec{u}_{\epsilon,\alpha}|^2$ converges to $|\vec{\nabla}\otimes\vec{u}|^2+\mu_1$ where μ_1 is a non-negative local measure, that $\vec{\nabla}.((|\vec{u}_{\epsilon,\alpha}|^2+2p_{\epsilon,\alpha})\vec{u}_{\epsilon,\alpha})$ converges to $\vec{\nabla}.((|\vec{u}|^2+2p)\vec{u})$ [since $\vec{u}$ is locally strongly convergent in $L^3_{t,x}$] so that finally $|\vec{u}_{\epsilon,\alpha}|^2\vec{\nabla}.\vec{u}_{\epsilon,\alpha}-2\alpha|\vec{u}_{\epsilon,\alpha}|^4+2p_{\epsilon,\alpha}\vec{\nabla}.\vec{u}_{\epsilon,\alpha}$ is bound to converge in $\mathcal{D}'$ to some distribution $-\mu_2$ where μ_2 is a non-negative local measure. This proves that the solution $\vec{u}$ of (1) is suitable. $\diamond$

Remark : A different equation has been considered by Plecháč and Šverák [PLE 03] in case of a radially symmetrical compactly supported initial data. They modify equation (59) into

$$\begin{cases} \partial_t\vec{u}_\epsilon + \vec{\nabla}.(\vec{u}_\epsilon\otimes\vec{u}_\epsilon) - \frac{1}{2}(\vec{\nabla}.\vec{u}_\epsilon)\vec{u}_\epsilon = \Delta\vec{u}_\epsilon - \vec{\nabla}p_\epsilon \\ \vec{u}_{\epsilon\ |t=0} = \vec{u}_0 \\ p_\epsilon = -\frac{1}{\epsilon}\vec{\nabla}.\vec{u}_\epsilon \end{cases} \tag{126}$$

When we write the energy balance (8) the remainder R_ϵ given by formula (9) is no longer equal to 0, since $\vec{u}_\epsilon$ is no longer divergence-free, but the bad term $|\vec{u}_\epsilon|^2\vec{\nabla}.\vec{u}_\epsilon$ has been removed. R_ϵ is just given by

$$R_\epsilon = 2p_\epsilon\vec{\nabla}.\vec{u}_\epsilon = -2\epsilon p_\epsilon^2 \leq 0. \tag{127}$$

However, we prefered to study the modified Vishik and Fursikov equations (76), since the damping ensures us that the solution $\vec{u}_{\epsilon,\alpha}$ is locally $L^4_tL^4_x$ so that we may use more easily the energy estimates.

6. Scaling and extractions.

The solutions to equations (76) in Theorems 8, 9 and 10 are constructed through a constant reiteration of the following Rellich compactness criterion [LEM 02] :

Lemma 1 :
If $T\in(0,+\infty)$ and if $(v_\theta)_{\theta>0}$ is a family of distributions on $(0,T)\times\mathbb{R}^3$ such that, for some positive s and σ, for every $\varphi\in\mathcal{D}'((0,T)\times\mathbb{R}^3)$ the family $(\varphi v_\theta)_{\theta>0}$ is bounded in $L^2_t((0,T),H^s_x)$ and the family $(\partial_t(\varphi v_\theta))_{\theta>0}$ is bounded in $L^2((0,T),H_x^{-\sigma})$, then there exists a sequence $(\theta_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to+\infty}\theta_n=0$ and such that v_{θ_n} converges to a distribution $v\in\mathcal{D}'((0,T)\times\mathbb{R}^3)$ and v_{θ_n} converges strongly to v in $L^2_tL^2_x$ norm on every compact subset of $(0,T)\times\mathbb{R}^3$.

Let us explain the way solutions to (76) are constructed when the initial data belongs to $\dot{M}^{2,3}$. Theorem 8 will give us a solution on $(0,T_\epsilon)\times\mathbb{R}^3$, where T_ϵ is controlled by below in a way that depends only on ϵ and on the norm of $\vec{u}_0$ in $\dot{M}^{2,3}$. We first try and construct a solution $\vec{u}_\gamma$ which we may control on $(0,\gamma^{-2}T_\epsilon)\times\mathbb{R}^3$ by considering equation (76) with initial data $\vec{u}_{0,\gamma}=\gamma^{-1}\vec{u}_0(\gamma^{-1}x)$; Theorem 8 gives a solution $\vec{v}_\gamma$ on $(0,T_\epsilon)\times\mathbb{R}^3$ associated to $\vec{u}_{0,\gamma}$, then by rescaling a solution $\vec{u}_\gamma=\gamma\vec{v}_\gamma(\gamma^2t,\gamma x)$ associated to $\vec{u}_0$ and defined on $(0,\gamma^{-2}T_\epsilon)\times\mathbb{R}^3$. If we want to use Lemma 1 to extract a global solution, we need to provide for every $R>1$ a uniform control of $\vec{u}_\gamma$ on every compact subset of $(0,RT_\epsilon)\times\mathbb{R}^3$ for $\gamma\leq R^{-2}$.

In oder to control $\vec{u}_\gamma$, we need to understand how $\vec{v}_\gamma$ is produced. We truncate $\vec{u}_{0,\gamma}$ by multiplication with a function $\theta(x/R)$ where $\theta\in\mathcal{D}'(\mathbb{R}^3)$ is equal to 1 on a neighbourhood of 0 then we project the truncated initial value on the solenoidal vector fields through Leray's projection operator and get $\vec{u}_{0,\gamma,R}=\mathbb{P}(\theta(x/R)\vec{u}_{0,\gamma})$. This is not a good way to process an initial value in L^2_{uloc} [BAS 06] since the Leray projection operator is not bounded on L^2_{uloc}, but this is a good way for an initial value in $\dot{M}^{2,3}$: we have $\|\vec{u}_{0,\gamma,R}\|_{\dot{M}^{2,3}}\leq C\|\vec{u}_0\|_{\dot{M}^{2,3}}$ for a constant C which doesn't depend on $\vec{u}_0$, γ nor R, and we have the *-weak convergence of $\vec{u}_{0,\gamma,R}$ to $\vec{u}_{0,\gamma}$. Theorem 8 gives a solution $\vec{v}_{\gamma,R}$ on $(0,T_\epsilon)\times\mathbb{R}^3$ associated to $\vec{u}_{0,\gamma,R}$ and we have indeed a uniform control of $\vec{v}_{\gamma,R}$ on every compact subset of $(0,T_\epsilon)\times\mathbb{R}^3$, this control depending only on $\|\vec{u}_{0,\gamma,R}\|_{\dot{M}^{2,3}}$ (and thus on $\|\vec{u}_0\|_{\dot{M}^{2,3}}$). If we look directly at the consequences of Theorem 8 however, this control does not appear precise enough to control $\vec{u}_\gamma$ uniformly with respect to γ : roughly speaking, we get a control of the $L^\infty_tL^2_x\cap L^2_tH^1_x$ norm of $\vec{v}_\gamma$ only on domains $(0,T_\epsilon)\times B(x_0,1)$ (with an estimate $O(1)$), hence of the $L^2_tL^2_x\cap L^2_tH^1_x$ norm of $\vec{u}_\gamma$ on domains $(0,\gamma^{-2}T_\epsilon)\times B(x_0,\gamma^{-1})$ with estimates $O(\gamma^{-1/2})$.

In order to get a better control on $\vec{u}_\gamma$, we need to explain more precisely how $\vec{v}_{\gamma,R}$ is produced. Following [LEL 10], we consider a mollified equation

$$(128_\eta) \quad \begin{cases} \partial_t \vec{v}_{\gamma,R,\eta} + \omega_\eta * \big((\vec{v}_{\gamma,R,\eta,\infty}.\vec{\nabla})\vec{v}_{\gamma,R,\eta,\infty}\big) = \Delta \vec{v}_{\gamma,R,\eta} - \alpha\omega_\eta * (|\vec{v}_{\gamma,R,\eta,\infty}|^2 \vec{v}_{\gamma,R,\eta,\infty}) - \vec{\nabla} p_{\gamma,R,\eta} \\ \vec{v}_{\gamma,R,\eta\ |t=0} = \vec{u}_{0,\gamma,R} \\ \vec{v}_{\gamma,R,\eta,\infty} = \omega_\eta * \vec{v}_{\gamma,R,\eta} \\ p_{\gamma,R,\eta} = -\frac{1}{\epsilon}\omega_\eta * \vec{\nabla}.\vec{v}_{\gamma,R,\eta,\infty} \end{cases}$$

where $\omega \in \mathcal{D}(\mathbb{R}^3)$ satisfies $\int \omega(x)\, dx = 1$ and where $\omega_\eta(x) = \eta^{-3}\omega(\eta^{-1}x)$. We get (through energy estimates) a control of the solution $\vec{v}_{\gamma,R,\eta}$ in $L^\infty_t L^2_x \cap L^2 \dot{H}^1_x$ norm uniformly with respect to $\eta > 0$ on $(0,+\infty) \times \mathbb{R}^3$. Lemma 1 then gives Theorem 8. Note that for equations (128_η) we have uniqueness of the solution.

We shall call $\mathcal{S}_{\gamma,R}$ the set of solutions $\vec{v}$ to (76) which are obtained from the initial value $\vec{u}_{0,\gamma,R}$ as a weak limit $\lim_{\eta_n\to 0} \vec{v}_{\gamma,R,\eta_n}$. Now, the precise mechanism of the proof is the following one : for $\mu > 0$, let us condider $\mu\vec{v}_{\gamma,R,\eta}(\mu^2 t, \mu x)$; equations (128_η) are no longer invariant through the rescaling, because of the convolutions with ω_η; we have to rescale ω_η into $\omega_{\eta/\mu}$; thus $\mu\vec{v}_{\gamma,R,\eta}(\mu^2 t, \mu x)$ is solution to equations $(128_{\eta/\mu})$ with initial value $\mu\vec{u}_{0,\gamma,R}(\mu x) = \vec{u}_{0,\gamma/\mu,R/\mu}(x)$. Thus, we find that

$$(129) \qquad \mu\vec{v}_{\gamma,R,\eta}(\mu^2 t, \mu x) = \vec{v}_{\gamma/\mu,R/\mu,\eta/\mu}(t,x)$$

Using another process of extractions, we find that for each $v \in \mathcal{S}_{\gamma,R}$ and every $\mu > 0$ there exists $\vec{w} \in \mathcal{S}_{\gamma/\mu,R/\mu}$ such that

$$(130) \qquad \mu\vec{v}(\mu^2 t, \mu x) = \vec{w}(t,x)$$

We note $Q_{T,x_0,\rho} = (0,T) \times B(x_0,\rho)$. We get through the proof of Theorem 9 [LEL 10] that

$$(131) \qquad \sup_{\gamma>0,R>0} \sup_{\vec{v}\in\mathcal{S}_{\gamma,R}} \sup_{x_0\in\mathbb{R}^3} \|\vec{v}\|_{L^\infty_t L^2_x(Q_{T_\epsilon,x_0,1})} + \|\vec{\nabla}\otimes\vec{v}\|_{L^2_t L^2_x(Q_{T_\epsilon,x_0,1})} \le C(\epsilon, \|\vec{u}_0\|_{\dot{M}^{2,3}}).$$

Using (130) and (131) we get

$$(132) \qquad \sup_{\gamma>0,R>0,\mu>0} \sup_{\vec{v}\in\mathcal{S}_{\gamma,R}} \sup_{x_0\in\mathbb{R}^3} \frac{\|\vec{v}\|_{L^\infty_t L^2_x(Q_{\mu^2 T_\epsilon,x_0,\mu})} + \|\vec{\nabla}\otimes\vec{v}\|_{L^2_t L^2_x(Q_{\mu^2 T_\epsilon,x_0,\mu})}}{\sqrt{\mu}} \le C(\epsilon, \|\vec{u}_0\|_{\dot{M}^{2,3}}).$$

Thus, we have for any $\gamma > 0$ and $R > 0$ a solution $\vec{v}_{\gamma,R}$ of (76) (with initial value $\vec{u}_{0,\gamma,R}$) on the strip $(0,T_\epsilon)\times\mathbb{R}^3$ with a uniform control given by (131) and (132) :

$$(133) \qquad \sup_{\gamma>0,R>0} \sup_{x_0\in\mathbb{R}^3} \|\vec{v}_{\gamma,R}\|_{L^\infty_t L^2_x(Q_{T_\epsilon,x_0,1})} + \|\vec{\nabla}\otimes\vec{v}_{\gamma,R}\|_{L^2_t L^2_x(Q_{T_\epsilon,x_0,1})} \le C(\epsilon, \|\vec{u}_0\|_{\dot{M}^{2,3}}).$$

and

$$(134) \qquad \sup_{\gamma>0,R>0,0<\mu\le 1} \sup_{x_0\in\mathbb{R}^3} \frac{\|\vec{v}_{\gamma,R}\|_{L^\infty_t L^2_x(Q_{\mu^2 T_\epsilon,x_0,\mu})} + \|\vec{\nabla}\otimes\vec{v}_{\gamma,R}\|_{L^2_t L^2_x(Q_{\mu^2 T_\epsilon,x_0,\mu})}}{\sqrt{\mu}} \le C(\epsilon, \|\vec{u}_0\|_{\dot{M}^{2,3}}).$$

Letting $1/R$ go to 0, we apply Lemma 1 and extract a weak limit $\vec{v}_\gamma$ on $(0,T_\epsilon) \times \mathbb{R}^3$ with strong local convergence in $L^2_t L^2_x$ norm. $\vec{v}_\gamma$ is a solution of (76) (with initial value $\vec{u}_{0,\gamma}$) on the strip $(0,T_\epsilon) \times \mathbb{R}^3$ and, from (133) and (134), we get

$$(135) \qquad \sup_{\gamma>0} \sup_{x_0\in\mathbb{R}^3} \|\vec{v}_\gamma\|_{L^\infty_t L^2_x(Q_{T_\epsilon,x_0,1})} + \|\vec{\nabla}\otimes\vec{v}_\gamma\|_{L^2_t L^2_x(Q_{T_\epsilon,x_0,1})} \le C(\epsilon, \|\vec{u}_0\|_{\dot{M}^{2,3}}).$$

and

$$(136) \qquad \sup_{\gamma>0,0<\mu\le 1} \sup_{x_0\in\mathbb{R}^3} \frac{\|\vec{v}_\gamma\|_{L^\infty_t L^2_x(Q_{\mu^2 T_\epsilon,x_0,\mu})} + \|\vec{\nabla}\otimes\vec{v}_\gamma\|_{L^2_t L^2_x(Q_{\mu^2 T_\epsilon,x_0,\mu})}}{\sqrt{\mu}} \le C(\epsilon, \|\vec{u}_0\|_{\dot{M}^{2,3}}).$$

Now, we rescale $\vec{v}_\gamma$ into $\vec{u}_\gamma$ by writing $\vec{u}_\gamma(t,x) = \gamma \vec{v}_\gamma(\gamma^2 t, \gamma x)$. $\vec{u}_\gamma$ is a solution of (76) (with initial value $\vec{u}_0$) on the strip $(0, \gamma^{-2}T_\epsilon) \times \mathbb{R}^3$ and, from (136), we get

$$\sup_{\gamma>0, 0<\mu\leq 1} \sup_{x_0 \in \mathbb{R}^3} \frac{\|\vec{u}_\gamma\|_{L^\infty_t L^2_x(Q_{\mu^2\gamma^{-2}T_\epsilon, x_0, \mu\gamma^{-1}})} + \|\vec{\nabla} \otimes \vec{u}_\gamma\|_{L^2_t L^2_x(Q_{\mu^2\gamma^{-2}T_\epsilon, x_0, \mu\gamma^{-1}})}}{\sqrt{\mu\gamma^{-1}}} \leq C(\epsilon, \|\vec{u}_0\|_{\dot{M}^{2,3}}). \tag{137}$$

Now, we let γ go to 0. We have uniform control for larger and larger domains as γ goes to 0; then, Lemma 1 and a diagonal extraction process allow us to extract a weak limit $\vec{u}$ on $(0, +\infty) \times \mathbb{R}^3$ with strong local convergence in $L^2_t L^2_x$ norm. $\vec{u}$ is a solution of (76) (with initial value $\vec{u}_0$) on the strip $(0, +\infty) \times \mathbb{R}^3$ and, from (133) and (137), we get

$$\sup_{R>0} \sup_{x_0 \in \mathbb{R}^3} \frac{\|\vec{u}\|_{L^\infty_t L^2_x(Q_{R^2 T_\epsilon, x_0, R})} + \|\vec{\nabla} \otimes \vec{u}_\gamma\|_{L^2_t L^2_x(Q_{R^2 T_\epsilon, x_0, R})}}{\sqrt{R}} \leq C(\epsilon, \|\vec{u}_0\|_{\dot{M}^{2,3}}). \tag{138}$$

Conclusion.

In the search of self-similar solutions $\vec{u}$ for equations (1), in the case of large initial data, a strategy could be to investigate the existence of self-similar solutions $\vec{u}_{\epsilon,\alpha}$ for equations (76). If we were able to prove that (76) has self-similar solutions for every $\epsilon > 0$ and $\alpha > 0$ (with $\epsilon < 4\alpha$), then Theorem 10 would give us a self-similar solution to (1). The small benefits we may find in studying (76) instead of (1) are the energy equality and the simple expression of the pressure. But the problem of finding large self-similar solutions of (76) is very similar to the problem of finding large self-similar solutions to (1) : the self-similarity precludes any contractivity argument (in contrast to the case of small initial data) thus any easy uniqueness argument to prove self-similarity.

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