Valuation and Hedging of CDS Counterparty Exposure in a Markov Copula Model

T. R. Bielecki [1,*] S. Crépey [2,4,†] M. Jeanblanc [2,3,4,†] B. Zargari [2,5]

[1] Illinois Institute of Technology, [2] Université d'Evry Val d'Essonne,
[3] Europlace Institute of Finance, [4] CRIS Consortium,
[5] Sharif University of Technology.

February 18, 2011

The authors thank S. Assefa and J.-P. Lardy for interesting discussions during the preparation of this article.

Abstract

A Markov model is constructed for studying the counterparty risk in a CDS contract. The 'wrong-way risk' in this model is accounted for by the possibility of the common default of the reference name and of the counterparty. A dynamic copula property as well as affine model specifications make pricing and calibration very efficient. We also consider the issue of dynamically hedging the CVA with a rolling CDS written on the counterparty. Numerical results are presented to show the adequacy of the behavior of CVA in the model with stylized features.

Keywords: Counterparty Credit Risk, CDS, CVA, Wrong-Way Risk, Dynamic Hedging.

1 Introduction

The sub-prime crisis has highlighted the importance of counterparty risk in OTC derivative markets, particularly in the case of credit derivatives. We consider in this paper the case of a Credit Default Swap with counterparty risk. This topic, which corresponds to the emblematic case of CDSs between Lehman and AIG, already received a lot of attention in

*The research of T.R. Bielecki was supported by NSF Grant DMS–0604789 and NSF Grant DMS–0908099.
†The research of this author benefited from the support of the DGE.

the literature. It can thus be considered as a benchmark problem of counterparty credit risk.

There has been a lot of research activity in the recent years devoted to valuation of counterparty risk. To quote but a few references:
• Huge and Lando [21] propose a rating-based approach,
• Hull and White [20] study this problem in the set-up of a static copula model,
• Jarrow and Yu [22] use an intensity contagion model, further considered in Leung and Kwok [26],
• Brigo and Chourdakis [11] work in the set-up of their Gaussian copula and CIR++ intensity model, extended to the issue of bilateral counterparty credit risk in Brigo and Capponi [10],
• Blanchet-Scalliet and Patras [8] or Lipton and Sepp [25] develop structural approaches,
• Stein and Lee [29] give a discussion of theoretical and practical issues regarding computations of credit valuation adjustment in the fixed income markets,
• Recent monographs of Cesari et al. [13] and Gregory [19] provide discussion of modeling and computational aspects regarding managing of exposure to counterparty risk.

In this paper (see [15] for a preparatory study in the set-up of a deterministic intensities model), we shall work in a Markovian copula set-up [5] with marginals auto-calibrated to the related CDS curves, the model dependence structure being determined by the possibility of simultaneous defaults of the counterparty and of the firm underlying the CDS. Here we apply the Markov copula approach to model joint default between counterparty and the reference name in a CDS contract. Exactly the same approach can be applied to modeling the "double-default" effect (cf. [2]).

Note that we limit ourselves to the so called *unilateral* counterparty risk, not considering the counterparty risk due to the possibility of 'one's own default'. Whether counterparty risk should be assessed on a unilateral or bilateral basis is a controversial issue. For discussion of bilateral counterparty risk we refer the reader to, for instance, Brigo and Capponi [10], Assefa et al. [1], or to Bielecki et al. [7].

There has been a lot of research activity in the recent years devoted to valuation of counterparty risk. In contrast, almost no attention has been devoted to quantitative studies of the problem of (dynamic) hedging of this form of risk. There is some discussion devoted to dynamic hedging of counterparty exposure in Cesari et al. [13] and in Gregory [19].

In this paper, we present formal mathematical results that provide analytical basis for the quantitative methodology of dynamic hedging of counterparty risk. Due to space limitation, we only provide a rather preliminary and incomplete study. But, we address the main theoretical issues of dynamic hedging of CVA, nevertheless. In particular, we provide formulae for mean-variance delta for a combined hedging of spread risk and jump-to-default risk, as well as a formula for mean-variance delta for hedging of the jump-to-default risk.

It needs to be stressed though that in this pilot study we only focus on hedging against exposure to the specific credit risk of the counterparty. So, in essence, we are only concerned with the credit deltas relative to the counterparty (sensitivities to counterparty spread, and counterparty jump-to-default). Credit deltas relative to the reference name, as well as the market deltas (specifically, sensitivities to rates), and also the so called cross-gammas, i.e.

the change in market sensitivities due to change in spread (cf. [14]), will be considered in a separate study.

1.1 Outline of the Paper

Section 2 recalls the basics of a CDS with counterparty credit risk. In Section 3, we present our Markov model. In Section 4, the main valuation results are derived. Hedging is discussed in Section 5. In Section 6, we propose an affine intensities specification of the model, and discuss its calibration. A variant of the model using extended CIR intensity processes is devised in Section 7. Section 8 presents numerical results.

2 Cash Flows and Pricing in a General Set-Up

In this section, we briefly recall the basics of CDS (unilateral) counterparty risk, referring the reader to [15] for every detail. Let us thus be given a CDS with maturity T and contractual spread κ, as considered from the perspective of the investor, assumed by default to be buyer of default protection on the reference firm of the CDS (case of payer CDS). In the case of a receiver CDS all the related quantities will be denoted with a 'bar', like $\overline{\text{CDS}}$.

Indices 1 and 2 will refer to quantities related to the firm and to the counterparty, respectively. For instance, τ_1 and τ_2 will denote their respective default times, whereas R_1 and R_2 will stand for the corresponding recoveries upon default.

The default times τ_1 and τ_2 cannot occur at fixed times, but may occur simultaneously. The recovery rates R_1 and R_2 are assumed to be constant for simplicity. Finally one assumes a deterministic discount factor $\beta(t) = \exp(-rt)$, for a constant short-term interest-rate function r. Given a risk neutral pricing model $(\Omega, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is a given filtration for which the τ_is are stopping times, let $\mathbb{E}_\theta$ stand for the conditional expectation under $\mathbb{P}$ given $\mathcal{F}_\theta$, for any stopping time θ. Let 'risky CDS' and 'risk-free CDS' respectively refer to a CDS with and without consideration of the counterparty risk.

Definition 2.1 **(i)** *The* price *process of a risk-free CDS is given by* $P_t = \mathbb{E}_t p^t$, *where the* discounted cumulative risk-free cash flows *on* $(t, T]$ *are given by*

$$\beta(t)p^t = -\kappa \int_t^{\tau_1\wedge T} \beta(s)ds + (1 - R_1)\beta(\tau_1)\mathbb{1}_{t<\tau_1<T} \ . \tag{1}$$

For $\overline{CDS}$, *the risk-free cash flows are* $\bar{p}^t = -p^t$ *and the corresponding price process is* $\bar{P}_t = \mathbb{E}_t[\bar{p}^t] = -P_t$.

(ii) *The* price *process of a risky CDS is given by* $\Pi_t = \mathbb{E}_t \pi^t$, *where the* discounted cumulative risky cash flows *on* $(t, T]$ *are given by*

$$\begin{aligned}\beta(t)\pi^t &= -\kappa \int_t^{\tau_1\wedge\tau_2\wedge T} \beta(s)ds + \beta(\tau_1)(1 - R_1)\mathbb{1}_{t<\tau_1<T}\left[\mathbb{1}_{\tau_1<\tau_2} + R_2\mathbb{1}_{\tau_1=\tau_2}\right] \\ &\quad +\beta(\tau_2)\mathbb{1}_{t<\tau_2<\tau_1\wedge T}\left[R_2 P^+_{\tau_2} - P^-_{\tau_2}\right] ,\end{aligned} \tag{2}$$

in which P^+ (resp. P^-) stands for the positive (resp. negative) part of P. The corresponding risky cumulative cash flows and price process for $\overline{CDS}$ are given by

$$\begin{aligned}\beta(t)\bar{\pi}^t &= \kappa \int_t^{\tau_1\wedge\tau_2\wedge T} \beta(s)ds - (1-R_1)\beta(\tau_1)\mathbb{1}_{t<\tau_1\leq\tau_2\wedge T} \\ &\quad +\beta(\tau_2)\mathbb{1}_{t<\tau_2<\tau_1\wedge T}\left[R_2 P_{\tau_2}^- - P_{\tau_2}^+\right] ,\end{aligned}$$

and $\bar{\Pi}_t = \mathbb{E}_t[\bar{\pi}^t]$.
(iii) *The Credit Valuation Adjustments are the processes defined by, for $t \in [0,T]$,*

$$\Theta_t = \mathbb{1}_{\{t<\tau_2\}}(P_t - \Pi_t) , \quad \bar{\Theta}_t = \mathbb{1}_{\{t<\tau_2\}}(\bar{P}_t - \bar{\Pi}_t) .$$

Note that $p^t = \bar{p}^t = P_t = \bar{P}_t = 0$ for $t \geq \tau_1 \wedge T$, and $\pi^t = \bar{\pi}^t = \Pi_t = \bar{\Pi}_t = \Theta_t = \bar{\Theta}_t = 0$ for $t \geq \tau_1 \wedge \tau_2 \wedge T$.

Proposition 2.1 (See [1, 12, 15]) *One has, on $\{t < \tau_2\}$,*

$$\beta(t)\Theta_t = \mathbb{1}_{\{t<\tau_2\}}\mathbb{E}_t\left[\beta(\tau_2)\xi_{(\tau_2)}\right] , \quad \beta(t)\bar{\Theta}_t = \mathbb{1}_{\{t<\tau_2\}}\mathbb{E}_t\left[\beta(\tau_2)\bar{\xi}_{(\tau_2)}\right] ,$$

for the $\mathcal{F}_{\tau_2}$-measurable Potential Future Exposures *(PFEs) defined by*

$$\begin{aligned}\xi_{(\tau_2)} &:= (1-R_2)\left(\mathbb{1}_{\tau_2<\tau_1\wedge T}P_{\tau_2}^+ + \mathbb{1}_{\tau_2=\tau_1<T}(1-R_1)\right) , \\ \bar{\xi}_{(\tau_2)} &:= (1-R_2)P_{\tau_2}^-\mathbb{1}_{\tau_2<\tau_1\wedge T} .\end{aligned}$$

Remark 2.2 A major issue in regard to counterparty credit risk is the so-called *wrong-way risk*. In general the wrong-way risk is understood as follows: Wrong-way risk occurs when exposure to a counterparty is adversely correlated with the credit quality of that counterparty. This can be rephrased as the risk that the value of the contract may be particularly high from the perspective of the other party at the moment of default of the counterparty. In our model the wrong-way risk is accounted for by the two terms in $\xi_{(\tau_2)}$ for the payer CDS: by the 'large' term $1 - R_1$, as well as by possibly large term $P_{\tau_2}^+$.

Roughly speaking, the wrong-way risk manifests itself in two regimes:

Regime 1:

$\mathbb{P}(\tau_2$ is small) is large, and $\mathbb{P}(\tau_2 < \tau_1)$ is large. So, indeed, it would be the (large) term $P_{\tau_2}^+$ that accounts for the wrong-way risk. [As an extreme case one could consider the case where $\tau_2 = \varepsilon > 0$ with probability nearly one, and $\tau_1 = \tau_2 + \varepsilon'$. Then, correlation is 1 between τ_1 and τ_2, and, clearly, $P_{\tau_2}^+$ is the only term in the exposure, and it is large. This is a clear cut case of wrong-way risk: the fact that $\tau_2 = \varepsilon$ with probability nearly one means that the counterparty is of low credit quality, and the exposure is large.]

Regime 2:

$\mathbb{P}(\tau_2$ is small) is large, and $\mathbb{P}(\tau_2 = \tau_1)$ is large. So, indeed, it would be the (large) term $(1 - R_1)$ that accounts for the wrong-way risk. [As an extreme case one could consider the case where $\tau_2 = \varepsilon > 0$ with probability nearly one, and $\tau_1 = \tau_2$. Here the only exposure term is $(1 - R_1)$. Again this is the clear cut case of the wrong-way risk.]

For the receiver CDS there is no wrong-way risk, at least not of this type.

3 Model

Let $\mathrm{H} = (H^1, H^2)$ denote the pair of the default indicator processes, so $H_t^i = \mathbb{1}_{\tau_i \le t}$. Given a *factor process* $\mathrm{X} = (X_1, X_2)$, to be made precise below, we consider a Markovian model of the pair (X, H) relative to its own filtration $\mathbb{F} = \mathbb{X} \vee \mathbb{H}$, with generator given by, for $u = u(t, x, e)$ with $t \in \mathbb{R}_+, x = (x_1, x_2) \in \mathbb{R}^2, e = (e_1, e_2) \in \{0, 1\}^2$:

$$\begin{aligned}\mathcal{A}u(t,x,e) = \partial_t u(t,x,e) &+ \sum_{1 \le i \le 2} l_i(t,x_i)\left(u(t,x,e^i) - u(t,x,e)\right) + l_3(t)\left(u(t,x,1,1) - u(t,x,e)\right)\\ &+ \sum_{1 \le i \le 2}\Big(b_i(t,x_i)\partial_{x_i}u(t,x,e) + \frac{1}{2}\sigma_i^2(t,x_i)\partial^2_{x_i^2}u(t,x,e)\Big) + \varrho\sigma_1(t,x_1)\sigma_2(t,x_2)\partial^2_{x_1,x_2}u(t,x,e)\ ,\end{aligned} \tag{3}$$

where, for $i = 1, 2$:
• e^i denotes the vector obtained from e, by replacing the component i by number one,
• b_i and σ_i^2 denote factor *drift* and *variance* functions, and l_i is an *individual default intensity* function,
• ϱ and $l_3(t)$ respectively stand for a *factor correlation* and a *joint defaults intensity* function. The choice $\varrho = 0$ will thus correspond to independent factor processes X^1 and X^2, whereas it is also possible to consider a *common factor process* $X^1 = X^2 = X$ by letting $b_1 = b_2$, $\sigma_1 = \sigma_2$, $X_0^1 = X_0^2$ and $\varrho = 1$.

The $\mathbb{F}$-*intensity-matrix function* of H (see, e.g., Bielecki and Rutkowski [4]) is thus given by the following 4×4 matrix $A(t, x)$, where the first to fourth rows (or columns) correspond to the four possible states $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$ of H_t :

$$A(t,x) = \begin{bmatrix} -l(t,x) & l_1(t,x_1) & l_2(t,x_2) & l_3(t) \\ 0 & -q_2(t,x_2) & 0 & q_2(t,x_2) \\ 0 & 0 & -q_1(t,x_1) & q_1(t,x_1) \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

with, for every $i = 1, 2$,

$$q_i(t, x_i) = l_i(t, x_i) + l_3(t) \tag{4}$$

and $l(t, x) = l_1(t, x_1) + l_2(t, x_2) + l_3(t)$. We assume standard regularity and growth assumptions on the coefficients of $\mathcal{A}$ so as to ensure well-posedness of the related martingale problem (see, e.g., Ethier and Kurtz [18]). One then has,

Proposition 3.1 **(i)** *For every $i = 1, 2$, the process (X^i, H^i) is an $\mathbb{F}$-Markov process with generator given by, for $u_i = u_i(t, x_i, e_i)$, with $t \in \mathbb{R}_+, x_i \in \mathbb{R}, e_i \in \{0, 1\}$:*

$$\begin{aligned}\mathcal{A}_i u_i(t, x_i, e_i) =& \partial_t u_i(t, x_i, e_i) + b_i(t, x_i)\partial_{x_i} u_i(t, x_i, e_i) + \frac{1}{2}\sigma_i^2(t, x_i)\partial^2_{x_i^2} u_i(t, x_i, e_i) \\ &+ q_i(t, x_i)\left(u_i(t, x_i, 1) - u_i(t, x_i, e_i)\right) .\end{aligned} \tag{5}$$

The $\mathbb{F}$-intensity matrix function of H^i is thus given by

$$A_i(t, x_i) = \begin{bmatrix} -q_i(t, x_i) & q_i(t, x_i) \\ 0 & 0 \end{bmatrix}$$

In other words, the process M^i defined by, for $i = 1, 2$,

$$M^i_t = H^i_t - \int_0^t (1 - H^i_s) q_i(s, X^i_s) ds \ , \tag{6}$$

is an $\mathbb{F}$-martingale.
(ii) *One has, for every $t \geq 0$,*

$$\mathbb{P}(\tau_i > t) = \mathbb{E} \exp\big(- \int_0^t q_i(u, X^i_u) du\big) \quad , \quad \mathbb{P}(\tau_1 \wedge \tau_2 > t) = \mathbb{E} \exp\big(- \int_0^t l(u, \mathrm{X}_u) du\big) \ . \tag{7}$$

Proof. **(i)** Applying the operator $\mathcal{A}$ in (3) to $u(t, x, e) := u_i(t, x_i, e_i)$, one gets,

$$\mathcal{A}u(t, x, e) = \mathcal{A}_i u_i(t, x_i, e_i),$$

where $\mathcal{A}_i$ is the operator defined in (5). In view of the Markov property of (X, H), the process $\mathcal{M}^i$ defined by

$$\mathcal{M}^i_t := u_i(t, X^i_t, H^i_t) - \int_0^t \mathcal{A}_i u_i(s, X^i_s, H^i_s) ds = u(t, \mathrm{X}_t, \mathrm{H}_t) - \int_0^t \mathcal{A}u(s, \mathrm{X}_s, \mathrm{H}_s) ds \, ,$$

is an $\mathbb{F}$-martingale. By the martingale characterization of Markov processes, the process (t, X^i, H^i) is thus $\mathbb{F}$-Markovian with generator $\mathcal{A}_i$. In particular for $u_i(t, x_i, e_i) := e_i$, one has $\mathcal{A}_i u_i(t, x_i, e_i) = q_i(t, x_i)(1 - e_i)$ and the martingale $\mathcal{M}^i$ coincides with M^i as of (6).
(ii) Since $\mathbb{P}(\tau_i > t) = \mathbb{E}\mathbb{1}_{H^i_t = 0}$ and $\mathbb{P}(\tau_1 \wedge \tau_2 > t) = \mathbb{E}\mathbb{1}_{H^1_t = H^2_t = 0}$, and in view of the Markov properties of (X^i, H^i) and (X, H), identities (7) can be checked by verification in the related Kolmogorov equations. □

In the terminology of [5], the model (X, H) is a Markovian copula model with marginals (X^i, H^i)s, or, in the common factor case $X^1 = X^2 = X$, marginals (X, H^i)s.

4 Pricing

Lemma 4.1 **(i)** *For every $i = 1, 2$ and function $p = p(t, x_i)$, one has, for $t \in [0, T]$,*

$$\mathbb{E}_t \int_t^T \beta(s)(1 - H^i_s) p(s, X^i_s) ds = (1 - H^i_t)\beta(t) v(t, X^i_t) \ , \tag{8}$$

for a function $v = v(t, x_i)$ solving the following pricing PDE:

$$\begin{cases} v(T, x_i) = 0\,, \quad x_i \in \mathbb{R} \\ \Big(\partial_t + b_i(t, x_i)\partial_{x_i} + \frac{1}{2}\sigma_i^2(t, x_i)\partial^2_{x_i^2}\Big)v(t, x_i) - \big(r + q_i(t, x_i)\big)v(t, x_i) + p(t, x_i) = 0\,, \\ \qquad t \in [0, T),\ x_i \in \mathbb{R}\,, \end{cases} \tag{9}$$

or, equivalently to (9),

$$v(t, x_i) = \mathbb{E}\Big(\int_t^T e^{-\int_t^s (r + q_i(\zeta, X^i_\zeta))d\zeta} p(s, X^i_s)ds \,\Big|\, X^i_t = x_i\Big)\,. \tag{10}$$

(ii) *For every function $\pi = \pi(t, x)$, one has, for $t \in [0, T]$,*

$$\mathbb{E}_t \int_t^T \beta(s)(1 - H^1_s)(1 - H^2_s)\pi(s, \mathrm{X}_s)ds = (1 - H^1_t)(1 - H^2_t)\beta(t)u(t, \mathrm{X}_t)\,,$$

for a function $u = u(t, x)$ solving the following pricing PDE:

$$\begin{cases} u(T, x) = 0\,, \quad x \in \mathbb{R}^2 \\ \Big(\partial_t + \sum_{1\le i\le 2}\big(b_i(t, x_i)\partial_{x_i} + \frac{1}{2}\sigma_i^2(t, x_i)\partial^2_{x_i^2}\big) + \varrho\sigma_1(t, x_1)\sigma_2(t, x_2)\partial^2_{x_1, x_2}\Big)u(t, x) \\ \qquad -\big(r + l(t, x)\big)u(t, x) + \pi(t, x) = 0\,, \quad t \in [0, T),\ x \in \mathbb{R}^2\,, \end{cases} \tag{11}$$

or, equivalently to (11),

$$u(t, x) = \mathbb{E}\Big(\int_t^T e^{-\int_t^s (r + l(\zeta, \mathrm{X}_\zeta))d\zeta}\pi(s, \mathrm{X}_s)ds \,\Big|\, \mathrm{X}_t = x\Big)\,.$$

Proof. **(i)** The Markov property of (X^1, H^1) stated at Proposition 3.1(i) implies (8). Moreover, in view of the form (5) of the generator of (X^1, H^1), the function v has to satisfy (9). From Feynman-Kač formula, one then obtains (10).
(ii) The result follows as in point (i), using the form (3) of the generator of (X, H). □

Remark 4.1 Validity of this result and the related proof are in fact subject to suitable regularity and growth assumptions on the data, including the coefficient functions p and π. The strength of these assumptions depends on the meaning in which a solution to the pricing equations is sought for. Since these kinds of technicalities are not the main issue of the present paper, we refer the reader to the literature in this regard (see, for instance, Karatzas and Shreve [23] for classical solutions).

Let further $H^{\{1\}}$, $H^{\{2\}}$ and $H^{\{1,2\}}$ stand for the indicator processes of a default of the firm alone, of the counterparty alone, and of a simultaneous default of the firm and the counterparty, respectively. So

$$H^{\{1,2\}} = [H^1, H^2]\,,\ \ H^{\{1\}} = H^1 - H^{\{1,2\}}\,,\ \ H^{\{2\}} = H^2 - H^{\{1,2\}}\,,$$

where $[H^1, H^2]_t = \mathbb{1}_{\tau_1 = \tau_2 \le t}$ stands for the quadratic covariation of the default indicator processes H^1 and H^2.

Lemma 4.2 *The $\mathbb{F}$-intensity of H^ι is of the form $q_\iota(t, \mathrm{X}_t, \mathrm{H}_t)$ for a suitable function $q_\iota(t, x, e)$ for every $\iota \in I = \{\{1\}, \{2\}, \{1, 2\}\}$, namely,*

$$
\begin{aligned}
q_{\{1\}}(t, x, e) &= \mathbb{1}_{e_1=0} \left(\mathbb{1}_{e_2=0} l_1(t, x_1) + \mathbb{1}_{e_2=1} q_1(t, x_1)\right) \\
q_{\{2\}}(t, x, e) &= \mathbb{1}_{e_2=0} \left(\mathbb{1}_{e_1=0} l_2(t, x_2) + \mathbb{1}_{e_1=1} q_2(t, x_2)\right) \\
q_{\{1,2\}}(t, x, e) &= \mathbb{1}_{e=(0,0)} l_3(t) \ .
\end{aligned}
$$

Put another way, for every $\iota \in I$, the process M^ι defined by,

$$
M_t^\iota = H_t^\iota - \int_0^t q_\iota(s, \mathrm{X}_s, \mathrm{H}_s) ds \ ,
$$

is an $\mathbb{F}$-martingale, where the intensity processes $q_\iota(t, X_t, H_t)$s are given by

$$
\begin{aligned}
q_{\{1\}}(t, \mathrm{X}_t, \mathrm{H}_t) &= (1 - H_t^1) \left((1 - H_t^2) l_1(t, X_t^1) + H_t^2 q_1(t, X_t^1)\right) \\
q_{\{2\}}(t, \mathrm{X}_t, \mathrm{H}_t) &= (1 - H_t^2) \left((1 - H_t^1) l_2(t, X_t^2) + H_t^1 q_2(t, X_t^2)\right) \\
q_{\{1,2\}}(t, \mathrm{X}_t, \mathrm{H}_t) &= (1 - H_t^1)(1 - H_t^2) l_3(t) \ .
\end{aligned}
$$

Proof. An application of the $\mathbb{F}$-local martingale characterization of the $\mathbb{F}$-Markov process (X, H) with generator $\mathcal{A}$ in (3) yields the $\mathbb{F}$-intensity γ of process $H^1 H^2$:

$$
\gamma_t = (1 - H_t^1) H_t^2 l_1(t, X_t^1) + (1 - H_t^2) H_t^1 l_2(t, X_t^2) + (1 - H_t^1 H_t^2) l_3(t).
$$

Using Proposition 3.1(i), one deduces the desired expression for the $\mathbb{F}$-intensity process of

$$
H^{\{1,2\}} = [H^1, H^2] = -\int_0^\cdot H_{t-}^1 dH_t^2 - \int_0^\cdot H_{t-}^2 dH_t^1 + H^1 H^2 \ ,
$$

and then the $\mathbb{F}$-intensity processes of $H^{\{i\}}$ is obtained using $H^{\{i\}} = H^i - H^{\{1,2\}}$, for $i = 1, 2$.
□

We are now in a position to derive the risk-free and risky CDS pricing equations.

Proposition 4.3 **(i)** *The price of the risk-free CDS admits the representation:*

$$
P_t = (1 - H_t^1) v(t, X_t^1) \ ,
$$

for a pre-default pricing function $v = v(t, x_1)$ as of Lemma 4.1(i), with $i = 1$ and

$$
p(t, x_1) = (1 - R_1) q_1(t, x_1) - \kappa \ . \tag{12}
$$

(ii) *The price of the risky CDS admits the representation:*

$$
\Pi_t = (1 - H_t^1)(1 - H_t^2) u(t, \mathrm{X}_t) \ ,
$$

for a pre-default pricing function $u = u(t, x)$ as of Lemma 4.1(ii) with

$$
\pi(t, x) = (1 - R_1)\big[l_1(t, x_1) + R_2 l_3(t)\big] + l_2(t, x_2)\big[R_2 v^+(t, x_1) - v^-(t, x_1)\big] - \kappa \ . \tag{13}
$$

(iii) *The price of the risky* $\overline{\text{CDS}}$ *admits the representation:*

$$\bar{\Pi}_t = (1 - H_t^1)(1 - H_t^2)\bar{u}(t, \mathrm{X}_t) ,$$

for a pre-default pricing function $\bar{u} = \bar{u}(t,x)$ *as of Lemma 4.1(ii) with*

$$\bar{\pi}(t,x) = \kappa - (1 - R_1)q_1(t,x_1) + l_2(t,x_2)\big[R_2 v^-(t,x_1) - v^+(t,x_1)\big] . \tag{14}$$

Proof. **(i)** One has $P_t = \mathbb{E}_t(p^t)$, with

$$\begin{aligned}\beta(t)p^t &= -\kappa \int_t^T \beta(s)(1 - H_s^1)ds + (1 - R_1)\int_t^T \beta(s)dH_s^1 \\ &= \int_t^T \beta(s)(1 - H_s^1)p(s, X_s^1)ds + (1 - R_1)\int_t^T \beta(s)dM_s^1 ,\end{aligned}$$

with p defined by (12). Since $\mathbb{E}_t(\int_t^T \beta(s)dM_s^1) = 0$, the result follows by an application of Lemma 4.1(i).
(ii) One has $\Pi_t = \mathbb{E}_t(\pi^t)$, with π^t defined in (2). As in part (i), we write π^t in terms of integrals with respect to H^ιs:

$$\begin{aligned}\beta(t)\pi^t \;=\; & -\kappa \int_t^T \beta(s)(1 - H_s^1)(1 - H_s^2)ds + (1 - R_1)\int_t^T \beta(s)(1 - H_{s-}^2)dH_s^{\{1\}} \\ & + R_2(1 - R_1)\int_t^T \beta(s)dH_s^{\{1,2\}} + \int_t^T \beta(s)\big[R_2 v^+(s, X_s^1) - v^-(s, X_s^1)\big](1 - H_{s-}^1)dH_s^{\{2\}} ,\end{aligned}$$

where (i) was used in the last term. But in view of Lemma 4.2, this expression coincides, up to a martingale term, with

$$\int_t^T \beta(s)(1 - H_s^1)(1 - H_s^2)\pi(s, \mathrm{X}_s)ds ,$$

where π is given by (13). The result then follows by an application of Lemma 4.1(ii).
(iii) can be proved similarly to (ii). □

Note finally that in the case of time-deterministic intensities, the valuation PDEs reduce to ODEs, and semi-explicit formulas in the form of integrals with respect to time can be obtained for most quantities of interest, including the CVAs (see [15]).

5 Hedging of Counterparty Exposure

In order to discuss hedging of counterparty exposure we introduce now the cumulative CVA process. We first focus on the payer CDS, the main results for $\overline{\text{CDS}}$ being then given in Proposition 5.6.

Definition 5.1 *The cumulative CVA process $\widehat{\Theta}$ is given as, for $t \in [0,T]$,*

$$\beta(t)\widehat{\Theta}_t = \beta(t \wedge \tau_2)\left(\widehat{P}_{t\wedge\tau_2} - \widehat{\Pi}_{t\wedge\tau_2}\right) ,$$

where $\widehat{P}$ (resp. $\widehat{\Pi}$) denotes the cumulative *risk-free (resp. risky) CDS price process such that $\beta(t)\widehat{P}_t = \mathbb{E}_t p^0$ (resp. $\beta(t)\widehat{\Pi}_t = \mathbb{E}_t \pi^0$).*

Here, we shall focus on the cumulative CVA as it is this process that enjoys the martingale properties (after discounting) which are important for the hedging endeavor. One has by application of Proposition 2.1:

$$\beta(t)\widehat{\Theta}_t = \mathbb{E}_t\big[\beta(\tau_2)\xi_{(\tau_2)}\big] .$$

Note that on the set $\{\tau_2 \leq T\}$, the random variable $\xi_{(\tau_2)}$ can be represented as the value at time $t = \tau_2$ of the process ξ defined by, for $t \in [0,T]$,

$$\begin{aligned}
\xi_t &= (1-R_2)\Big(P_t^+ + (H_t^1 - H_{t-}^1)(1-R_1)\Big) \\
&= (1-R_2)\left((1-H_t^1)v^+(t,X_t^1) + (H_t^1 - H_{t-}^1)(1-R_1)\right) ,
\end{aligned} \tag{15}$$

where the second equality follows from Proposition 4.3.

In this section, we set henceforth $\beta = 1$ (null interest rate) for notational simplicity.

5.1 Dynamics of Cumulative CVA

The first step consists in deriving dynamics of the cumulative CVA. We start with the following elementary result,

Lemma 5.1 *For any $t \in (0,T]$, we have*

$$\begin{aligned}
d\widehat{\Theta}_t &= (1-H_{t-}^2)(d\widehat{P}_t - d\widehat{\Pi}_t) \\
&= (1-H_t^2)(dP_t - d\Pi_t) + (\Delta\widehat{P}_{\tau_2} - \Delta\widehat{\Pi}_{\tau_2})dH_t^2 \\
&= (1-H_t^2)(dP_t - d\Pi_t) + (\xi_{\tau_2} - \widehat{\Theta}_{\tau_2 -})dH_t^2 \\
&= (1-H_t^2)(dP_t - d\Pi_t) + (\xi_t - \widehat{\Theta}_{t-})dH_t^2 \\
&= (1-H_t^2)(dP_t - d\Pi_t) + (\xi_t - \Theta_{t-})dH_t^2.
\end{aligned} \tag{16}$$

Proof. The first line holds by definition of $\widehat{\Theta}$ and by application of Itô's formula. The second one follows from the fact that $p^0 - p^t = \pi^0 - \pi^t$ for any $t < \tau_2$. The remaining three equalities follow easily. □

Equation (16) is the key to hedging of counterparty risk. The dynamics of $\widehat{\Theta}$ splits into the "pre-counter-party-default" part $(1-H_t^2)(dP_t - d\Pi_t)$, and the "at-counter-party-default" part $(\xi_t - \widehat{\Theta}_{t-})dH_t^2$.

Specification of the above dynamics, that is specification of all martingale terms, is not easy in general. We now provide exact formulae for these coefficients, in our stylized Markovian copula model.

5.1.1 Markovian Case

We are in position now to particularize dynamics of the cumulative CVA for the model of Section 3.

Proposition 5.2 *For any $t \in [0,T]$, we have:*

$$\begin{aligned} d\,\widehat{\Theta}_t &= -(1-H^2_{t-})(v-u)dM^1_t \\ &\quad + \left\{(1-R_2)(1-H^1_{t-})v^+ - (1-H^1_{t-})(v-(1-H^2_{t-})u\right\} dM^2_t \\ &\quad +(v-u+(1-R_2)(1-R_1)-(1-R_2)v^+)dM^{\{1,2\}}_t \\ &\quad +(1-H^1_t)(1-H^2_t)((\partial_{x_1}v-\partial_{x_1}u)\sigma_1 dW^1_t - (\partial_{x_2}u)\sigma_2 dW^2_t)\,. \end{aligned}$$

where u and v stand for $u(t,\mathrm{X}_t)$ and $v(t,X^1_t)$.

Proof. Recall (16). Using Proposition 4.3 we obtain, for $0 \le t \le T$,

$$\begin{aligned} (1-H^2_t)dP_t &= (1-H^2_t)d\big((1-H^1_t)v(t,X^1_t)\big) \\ &= (1-H^1_t)(1-H^2_t)(\partial_{x_1}v)\sigma_1 dW^1_t - (1-H^2_{t-})v dM^1_t \\ &\quad +(H^2_t-H^2_{t-})v dH^1_t - (1-H^1_t)(1-H^2_t)((1-R_1)q_1-\kappa)dt\,, \end{aligned}$$

where the term $(1-H^2_{t-})v dM^1_t$ defines a martingale. In the same way,

$$\begin{aligned} (1-H^2_t)d\Pi_t &= (1-H^2_t)d\big((1-H^1_t)(1-H^2_t)u(t,X^1_t,X^2_t)\big) \\ &= -(1-H^2_{t-})u dM^1_t + (H^2_t-H^2_{t-})u dH^1_t \\ &\quad + (1-H^1_t)(1-H^2_t)(l_2 u-(1-R_1)(l_1+R_2 l_3)-l_2(R_2 v^+-v^-)+\kappa)dt \\ &\quad + (1-H^1_t)(1-H^2_t)((\partial_{x_1}u)\sigma_1 dW^1_t + (\partial_{x_2}u)\sigma_2 dW^2_t), \end{aligned}$$

and

$$\begin{aligned} (\xi_t-\widehat{\Theta}_{t-})dH^2_t &= \Big\{(1-R_2)\big((1-H^1_t)v^+ + (H^1_t-H^1_{t-})(1-R_1)\big) \\ &\quad -(1-H^1_{t-})v + (1-H^1_{t-})(1-H^2_{t-})u\Big\}\, dH^2_t. \end{aligned}$$

This, combined with Lemma 5.1, leads to

$$\begin{aligned} d\,\widehat{\Theta}_t &= -(1-H^2_{t-})(v-u)dM^1_t + (H^2_t-H^2_{t-})(v-u)dH^1_t \\ &\quad +(1-H^1_t)(1-H^2_t)((\partial_{x_1}v-\partial_{x_1}u)\sigma_1 dW^1_t - (\partial_{x_2}u)\sigma_2 dW^2_t) \\ &\quad -(1-H^1_t)(1-H^2_t)\Big(((1-R_1)q_1-\kappa)+(l_2 u-(1-R_1)(l_1+R_2 l_3)-l_2(R_2 v^+-v^-)+\kappa)\Big)dt \\ &\quad +\Big\{(1-R_2)\big((1-H^1_t)v^+ + (H^1_t-H^1_{t-})(1-R_1)\big) - (1-H^1_{t-})v + (1-H^1_{t-})(1-H^2_{t-})u\Big\}\, dH^2_t\,. \end{aligned}$$

We now observe that

$$(H^2_t-H^2_{t-})(v-u)dH^1_t = (v-u)dH^{\{1,2\}}_t\,,\ \ (1-H^1_t)v^+dH^1_t = (1-H^1_{t-})v^+dH^1_t - v^+dH^{\{1,2\}}_t\,.$$

Consequently,

$$\begin{aligned}
d\widehat{\Theta}_t &= -(1-H^2_{t-})(v-u)dM^1_t + (1-H^1_t)(1-H^2_t)\Big((\partial_{x_1}v-\partial_{x_1}u)\sigma_1 dW^1_t - (\partial_{x_2}u)\sigma_2 dW^2_t\Big) \\
&\quad + \Big(v-u+(1-R_1)(1-R_2)-(1-R_2)v^+\Big)dH^{\{1,2\}}_t \\
&\quad - (1-H^1_t)(1-H^2_t)\Big((1-R_1)\big(q_1-l_1-R_2 l_3\big) - l_2\big(v-u+(R_2-1)v^+\big)\Big)dt \\
&\quad + \Big\{(1-R_2)(1-H^1_{t-})v^+ - (1-H^1_{t-})(v-(1-H^2_{t-})u)\Big\}dH^2_t.
\end{aligned}$$

Now, using

$$dH^{\{1,2\}}_t = dM^{\{1,2\}}_t + l_3(1-H^1_t)(1-H^2_t)dt \quad , \quad dH^2 = dM^2_t + q_2(1-H^2_t)dt \ ,$$

the last expression can be rewritten as

$$\begin{aligned}
d\widehat{\Theta}_t &= -(1-H^2_{t-})(v-u)dM^1_t + (1-H^1_t)(1-H^2_t)\Big((\partial_{x_1}v-\partial_{x_1}u)\sigma_1 dW^1_t - (\partial_{x_2}u)\sigma_2 dW^2_t\Big) \\
&\quad + \Big(v-u+(1-R_1)(1-R_2)-(1-R_2)v^+\Big)dM^{\{1,2\}}_t \\
&\quad + \Big\{(1-R_2)(1-H^1_{t-})v^+ - (1-H^1_{t-})(v-(1-H^2_{t-})u\Big\}dM^2_t \\
&\quad + (1-H^1_t)(1-H^2_t)\Big\{\big(v-u+(1-R_1)(1-R_2)-(1-R_2)v^+\big)l_3 \\
&\qquad\qquad - (1-R_1)\big(q_1-l_1-R_2 l_3\big) + l_2(v-u+(R_2-1)v^+) \\
&\qquad\qquad + \big((1-R_2)v^+ - v + u\big)q_2\Big\}dt \ ,
\end{aligned}$$

where the dt-coefficient simplifies to zero, which proves the result. □

As a corollary, we obtain the following representation for the dynamics of CVA, which will be used in Section 5.2,

Corollary 5.3 *For any $t \in [0,T]$ we have*

$$\begin{aligned}
d\,\widehat{\Theta}_t &= -(1-H^1_{t-})(1-H^2_{t-})(v-u)dM^{\{1\}}_t \\
&\quad + (1-H^1_{t-})(1-H^2_{t-})\Big((1-R_2)v^+ - (v-u)\Big)dM^{\{2\}}_t \\
&\quad + (1-H^1_{t-})(1-H^2_{t-})\Big((1-R_1)(1-R_2) - (v-u)\Big)dM^{\{1,2\}}_t \\
&\quad + (1-H^1_t)(1-H^2_t)((\partial_{x_1}v-\partial_{x_1}u)\sigma_1 dW^1_t - (\partial_{x_2}u)\sigma_2 dW^2_t) \\
&=: \widetilde{\eta}^{\{1\}}_t dM^{\{1\}}_t + \widetilde{\eta}^{\{2\}}_t dM^{\{2\}}_t + \widetilde{\eta}^{\{1,2\}}_t dM^{\{1,2\}}_t + \widetilde{\gamma}^1_t dW^1_t + \widetilde{\gamma}^2_t dW^2_t \ . \qquad (17)
\end{aligned}$$

Recall that if $R_2 = 1$ then $v = u$ so that in this case $\widetilde{\eta}^{\{1\}} = \widetilde{\eta}^{\{2\}} = \widetilde{\eta}^{\{1,2\}} = \widetilde{\gamma}^1 = \widetilde{\gamma}^2 = 0$. In particular, $\Theta_t = \widehat{\Theta}_t = 0$ for all t. Now, using the convention that $\frac{0}{0} = 1$ we can write

$$d\,\widehat{\Theta}_t =: (1-R_2)\Big(\eta^{\{1\}}_t dM^{\{1\}}_t + \eta^{\{2\}}_t dM^{\{2\}}_t + \eta^{\{1,2\}}_t dM^{\{1,2\}}_t + \gamma^1_t dW^1_t + \gamma^2_t dW^2_t\Big) \ , \qquad (18)$$

where $\eta^{\{1\}} = \frac{\widetilde{\eta}^{\{1\}}}{1-R_2}$, and accordingly for the remaining coefficients.

5.2 Hedging of CVA

We shall apply the above results to the problem of dynamically hedging the CDS counterparty risk in our Markovian copula model.

As it is seen from Corollary 5.3 there are five martingale terms to hedge in the dynamics for $\widehat{\Theta}$. Three of them, namely those in $dM_t^{\{1,2\}}, dM_t^{\{2\}}$ and dW_t^2, induce the risk directly related to the counterparty. In this paper, we are only concerned with hedging of these three terms in the mean variance sense, using a single hedging instrument taken as a rolling CDS contract written on the counterparty. Of course we also implicitly trade in the savings account (which is worth a constant, in the present nil interest rates set-up) for the purpose of making the strategy self-financing.

5.2.1 Rolling CDS

For the purpose of dynamically hedging the CVA on the CDS on name one, we shall now consider a *rolling CDS*[1] referencing the counterparty, that is corresponding to the default time τ_2. The concept of the rolling CDS contract was formally introduced and studied in [6]. A rolling CDS is an 'abstract' contract which at any time t has similar features as the T-maturity CDS issued at this date t, in particular, its ex-dividend price is equal to zero at every t. Intuitively, one can think of the rolling CDS of a constant maturity T as a stream of CDSs of constant maturities equal to T that are continuously entered into and immediately unwound. Thus, a rolling CDS contract is equivalent to a self-financing trading strategy that at any given time t enters into a CDS contract of maturity T and then unwinds the contract at time $t + dt$.

Remark 5.2 We shall use here a simplifying assumption that the recovery R_2 is generic, that is, it is the same for all CDS contracts referencing the same default τ_2 and with the same maturity T. Otherwise, for every fixed maturity date T, we would need to consider the whole class of protection payment processes, indexed by the initiation date.

The main result regarding the dynamics of the mark-to-market value of a rolling CDS contract is the following

Lemma 5.4 ([6]) *The cumulative value process, say $\widehat{Q}$, of a rolling CDS referencing the counterparty, is an $\mathbb{F}$-martingale, and its dynamics are given as*

$$\begin{aligned} d\widehat{Q}_t &= (1-H_t^2)\left((1-R_2)\partial_x g(t,X_t^2) - \kappa(t,X_t^2)\partial_x f(t,X_t^2)\right)\sigma_2(t,X_t^2)dW_t^2 \;+\; (1-R_2)dM_t^2 \\ &=: (1-R_2)\left((1-H_t^2)\psi_t dW_t^2 + dM_t^{\{2\}} + dM_t^{\{1,2\}}\right) \end{aligned} \tag{19}$$

where g and f denote the pre-default pricing functions of the unit protection and fee legs of

[1] It is clear that for the hedging purposes one needs to use rolling CDS contracts, which are entered into with counterparties that are remote from default.

the ordinary CDS contract initiated at time t, so that

$$f(t,X_t^2)=\mathbb{E}\left(\int_t^T e^{-\int_t^u q_2(v,X_v^2)dv}du \,|\, X_t^2\right) ,$$

$$g(t,X_t^2)=\mathbb{E}\left(\int_t^T e^{-\int_t^u q_2(v,X_v^2)dv}q_2(u,X_u^2)du \,|\, X_t^2\right) ,$$

and $\kappa=(1-R_2)g/f$ is the corresponding CDS fair spread function.

5.2.2 Mean-variance Hedging

In principle, given that one has at one's disposal sufficiently many liquid traded instruments, one can dynamically replicate all risk sources that show up in the CVA in (17), namely $M^{\{1\}}$, $M^{\{2\}}$, $M^{\{1,2\}}$, W^1 and W^2.

In this paper we shall not discuss this dynamic replication though, but rather we shall focus on mean-variance hedging of CVA, using the rolling CDS on the counterparty (along with the savings account) as hedging instrument. Let thus ζ be a real-valued process, representing the number of units of rolling CDS which are held in a self-financing hedging strategy of CVA. Invoking (17) and (19) we conclude that the tracking error process e_t of the hedged portfolio satisfies, for $t\in[0,T]$,

$$\begin{aligned} de_t &= d\widehat{\Theta}_t-(1-H^1_{t-})\zeta_t d\widehat{Q}_t \\ \frac{de_t}{1-R_2} &= \eta_t^{\{1\}}dM_t^{\{1\}}+(\eta_t^{\{1,2\}}-\zeta_t)dM_t^{\{1,2\}}+(\eta_t^{\{2\}}-(1-H^1_{t-})\zeta_t)dM_t^{\{2\}} \\ &\quad +\gamma_t^1 dW_t^1+(\gamma_t^2-\zeta_t\psi_t^2)dW_t^2 \ . \end{aligned}$$

where the η and γ's were defined in (17) and ψ in (19). We thus obtain the following result,

Proposition 5.5 *For a payer CDS:* **(i)** *the self-financing strategy that minimizes the risk neutral variance of the tracking error is given, on the set $\{t\le\tau_1\wedge\tau_2\}$, as*

$$\begin{aligned} \zeta_t^{va} &= \frac{\eta_t^{\{2\}}d\langle M^{\{2\}}\rangle_t+\eta_t^{\{1,2\}}d\langle M^{\{1,2\}}\rangle_t+(\varrho\gamma_t^1+\gamma_t^2)\psi_t dt}{d\langle M^{\{2\}}\rangle_t+d\langle M^{\{1,2\}}\rangle_t+\psi_t^2 dt} \\ &= \frac{l_2(t,X_t^2)\eta_t^{\{2\}}+l_3(t)\eta_t^{\{1,2\}}+(\varrho\gamma_t^1+\gamma_t^2)\psi_t}{l_2(t,X_t^2)+l_3(t)+\psi_t^2} \ ; \end{aligned}$$

(ii) *The self-financing strategy that minimizes the risk neutral variance of the jump–to–counterparty–default risk is given, on the set $\{t\le\tau_1\wedge\tau_2\}$, as*

$$\begin{aligned} \zeta_t^{jd} &= \frac{l_2(t,X_t^2)\eta_t^{\{2\}}+l_3(t)\eta_t^{\{1,2\}}}{l_2(t,X_t^2)+l_3(t)} \\ &= \left(P_t^+-\frac{\Theta_{t-}}{1-R_2}\right)\frac{l_2(t,X_t^2)}{l_2(t,X_t^2)+l_3(t)}+\left((1-R_1)-\frac{\Theta_{t-}}{1-R_2}\right)\frac{l_3(t)}{l_2(t,X_t^2)+l_3(t)} \\ &= \frac{1}{1-R_2}\left(\mathbb{E}(\xi\,|\,\mathcal{F}_{\tau_2-})\,|_{\tau_2=t}-\Theta_{t-}\right) \ . \end{aligned}$$

The ζ^{jd} hedging strategy thus changes the counterparty jump-to-default exposure from ξ to $\mathbb{E}(\xi \,|\, \mathcal{F}_{\tau_2 -})\,|_{\tau_2 = t}$, the 'best guess' of ξ available right before τ_2.

Remark 5.3 Since τ_2 is an $\mathbb{F}$-stopping time, the $\mathcal{F}_{\tau_2-}$-measurable random variable $\mathbb{E}\left(\xi|\mathcal{F}_{\tau_2-}\right)$ can be represented as Y_{τ_2}, for some $\mathbb{F}$-predictable process Y (see Dellacherie and Meyer [16], Thm 67.b). In the above proposition, we denote $\mathbb{E}\left(\xi|\mathcal{F}_{\tau_2-}\right)|_{\tau_2=t} = Y_t$. A similar remark applies to the notation $\mathbb{E}\left(\bar{\xi}|\mathcal{F}_{\tau_2-}\right)|_{\tau_2=t} = \bar{Y}_t$ in Proposition 5.6 below.

In the case of the receiver CDS, let

$$
\begin{aligned}
\bar{\eta}_t^{\{1\}} &= (1-H^1_{t-})(1-H^2_{t-})(v+\bar{u}) \\
\bar{\eta}_t^{\{2\}} &= (1-H^1_{t-})(1-H^2_{t-})\Big((1-R_2)v^- + (v+\bar{u})\Big) \\
\bar{\eta}_t^{\{1,2\}} &= (1-H^1_{t-})(1-H^2_{t-})(v+\bar{u}) \\
\bar{\gamma}_t^1 &= -(1-H^1_t)(1-H^2_t)\sigma_1(\partial_{x_1} v + \partial_{x_1}\bar{u}) \\
\bar{\gamma}_t^2 &= -(1-H^1_t)(1-H^2_t)\sigma_2\partial_{x_2}\bar{u}\ ,
\end{aligned}
$$

Proposition 5.6 *For a receiver CDS:* **(i)** *The self-financing strategy that minimizes the risk neutral variance of the tracking error is given, on the set $\{t \le \tau_1 \wedge \tau_2\}$, as*

$$
\begin{aligned}
\bar{\zeta}_t^{va} &= \frac{\bar{\eta}_t^{\{2\}} d\langle M^{\{2\}}\rangle_t + \bar{\eta}_t^{\{1,2\}} d\langle M^{\{1,2\}}\rangle_t + (\varrho\gamma_t^1 + \gamma_t^2)\bar{\psi}_t dt}{d\langle M^{\{2\}}\rangle_t + d\langle M^{\{1,2\}}\rangle_t + \bar{\psi}_t^2 dt} \\
&= \frac{l_2(t, X_t^2)\bar{\eta}_t^{\{2\}} + l_3(t)\bar{\eta}_t^{\{1,2\}} + (\varrho\gamma_t^1 + \gamma_t^2)\bar{\psi}_t}{l_2(t, X_t^2) + l_3(t) + \bar{\psi}_t^2}\ ;
\end{aligned}
$$

(ii) *The self-financing strategy that minimizes the risk neutral variance of the jump–to–counterparty–default risk is given, on the set $\{t \le \tau_1 \wedge \tau_2\}$, as*

$$
\begin{aligned}
\bar{\zeta}_t^{jd} &= \frac{l_2(t, X_t^2)\bar{\eta}_t^{\{2\}} + l_3(t)\bar{\eta}_t^{\{1,2\}}}{l_2(t, X_t^2) + l_3(t)} = \left(P_t^- - \frac{\bar{\Theta}_{t-}}{1-R_2}\right)\frac{l_2(t, X_t^2)}{l_2(t, X_t^2) + l_3(t)} \\
&= \frac{1}{1-R_2}\left(\mathbb{E}(\bar{\xi}\,|\,\mathcal{F}_{\tau_2-})\,|_{\tau_2=t} - \bar{\Theta}_{t-}\right)\ .
\end{aligned}
$$

The $\bar{\zeta}^{jd}$ hedging strategy thus changes the counterparty jump-to-default exposure from $\bar{\xi}$ to $\mathbb{E}(\bar{\xi} \,|\, \mathcal{F}_{\tau_2 -})\,|_{\tau_2 = t}$, the 'best guess' of $\bar{\xi}$ available right before τ_2.

6 Model Implementation

In view of the model generator (3), the model primitives are the factor coefficients b and σ and the intensity functions l_is for $i = 1$ to 3, or, equivalently to the latter via (4), the marginal intensity functions $q_1 = q_1(t, x_1)$ and $q_2 = q_2(t, x_2)$ and the joint defaults intensity function $l_3 = l_3(t)$. In this section, following the lines of Brigo et al. [10, 11], we shall specify the factors in the form of CIR++ processes. Let thus the X^is be affine processes of the form

$$dX_t^i = \eta(\mu_i - X_t^i)dt + \nu\sqrt{X_t^i}dW_t^i \;,$$

for non-negative coefficients η, μ_i and ν. One then sets

$$q_i(t, x_i) = f_i(t) + \delta x_i \;, \tag{20}$$

for functions $f_i(t)$ such that $f_i(t) \geq l_3(t)$ and $\delta \in \{0, 1\}$.

Remark 6.1 (ii) As in Brigo et al. [10, 11], we shall not restrict ourselves to the inaccessible origin case $2\eta\mu_i > \nu^2$, in order not to limit the range of the model CDS implied volatility.[2]
(ii) The restriction $f_i(t) \geq l_3(t)$ is imposed to guarantee that, consistently with (4), $q_i(t, X_t^i)$ defined by (20) is never smaller than $l_3(t)$.

In the sequel, by (2F), we mean the parametrization (20) with $\delta = 1$ and independent affine factors X^1 and X^2, that is two independent CIR++ factors. Also, we denote by (0F), the parametrization (20) with $\delta = 0$, that is without stochastic factors (case of time-deterministic, piecewise constant intensities).

6.1 Marginals

Under the model specification (20), one can derive a more explicit formula for the pricing function $v = v(t, x_1)$. Let $F_1(t) = \int_0^t f_1(s)ds$.

Proposition 6.1 *Assuming* (20), *one has*

$$\beta(t)v(t, x_1) = \int_t^T \beta(s)\big((1 - R_1)\mathcal{D}_\delta(s, t, x_1) - \kappa\big)\mathcal{E}_\delta(s, t, x_1)ds \;,$$

where we set, for $s \geq t$,

$$\begin{aligned} \mathcal{E}_\delta(s, t, x_1) &= \exp\Big(- \big(F_1(s) - F_1(t) + \delta\phi(s - t, 0)x_1 + \delta\xi(s - t, 0)\mu_1\big)\Big) \;, \\ \mathcal{D}_\delta(s, t, x_1) &= f_1(s) + \delta\eta\mu_1\phi(s - t, 0) + \delta\big(- \eta\phi(s - t, 0) - \frac{1}{2}\nu^2(\phi(s - t, 0))^2 + 1\big)x_1 \;, \end{aligned}$$

in which the functions ϕ and ξ are those of Lemma A.1.

Proof. Recall from Proposition 4.3 that

$$\beta(t)v(t, x_1) = \textstyle\int_t^T \beta(s)\mathbb{E}\big(e^{-\int_t^s q_1(\zeta, X_\zeta^1)d\zeta}p(s, X_s^1)|X_t^1 = x_1\big)ds$$

with

$$p(s, X_s^1) = (1 - R_1)q_1(s, X_s^1) - \kappa \;,\;\; q_1(t, X_t^1) = f_1(t) + \delta X_t^1 \;.$$

[2] Indeed in our numerical tests the calibrated parameters do not always satisfy $2\eta\mu_i > \nu^2$.

For $\delta = 0$, the result follows immediately and for $\delta = 1$, it is obtained by an application of Lemma A.1. $\square$

In particular the model break-even spread at time 0 of a risk-free CDS of maturity T on the firm, is given by

$$\kappa_0(T) = (1 - R_1)\frac{\int_0^T \beta(s)\mathcal{D}_\delta(s, 0, X_0^1)\mathcal{E}_\delta(s, 0, X_0^1)ds}{\int_0^T \beta(s)\mathcal{E}_\delta(s, 0, X_0^1)ds} \ .$$

We denote by p_1 the cumulative distribution function (c.d.f. hereafter) of τ_1, namely,

$$p_1(t) := \mathbb{P}(\tau_1 \leq t) = 1 - \mathcal{E}_\delta(t, 0, X_0^1) \ . \tag{21}$$

In the same way, one can obtain semi-explicit formulae for the forward spread and CDS option price. As we will see in Section 8, such formulae are useful while computing the associated implied volatility.

For $0 \leq T_a < T_b$, the forward spread (at time 0) of a CDS issued at time T_a with maturity T_b is given by

$$\kappa_0(T_a, T_b) = (1 - R_1)\frac{\int_{T_a}^{T_b} e^{-rs}\mathcal{D}_\delta(s, 0, X_0^1)\mathcal{E}_\delta(s, 0, X_0^1)ds}{\int_{T_a}^{T_b} e^{-rs}\mathcal{E}_\delta(s, 0, X_0^1)ds} \ . \tag{22}$$

Now we consider a payer (resp. receiver) CDS option which gives the right to enter at time T_a a payer (resp. receiver) CDS with maturity T_b and the contractual spread κ.[3] Then the price at time 0 of these CDS options are given by

$$e^{-rT_a}\mathbb{E}\big[(1 - H_{T_a}^1)v(T_a, X_{T_a}^1)^+\big] \text{ for a payer CDS option ,} \tag{23}$$

$$e^{-rT_a}\mathbb{E}\big[(1 - H_{T_a}^1)v(T_a, X_{T_a}^1)^-\big] \text{ for a receiver CDS option ,} \tag{24}$$

where $v(t, x_1)$ is the CDS price function as in Proposition 6.1.

Of course analogous formulae hold for a risk-free CDS and CDS options on the counterparty.

6.2 Joint Defaults

In case market prices of instrument sensitive to the dependence structure of default times are available (basket credit instrument on the firm and the counterparty), these can be used to calibrate l_3. Admittedly however, this situation is an exception rather than the rule. It is thus important to devise a practical way of calibrating l_3 in case such market data are not available.

Note that under parameterizations (0F) and (2F), one has

$$\mathbb{P}(\tau_1 > t, \tau_2 > t) = \mathbb{P}(\tau_1 > t)\mathbb{P}(\tau_2 > t)e^{L_3(t)}, \tag{25}$$

[3] To avoid ambiguity, we call this contractual spread ,"the strike".

for $L_3(t) = \int_0^t l_3(s)ds$. A possible procedure thus consists in 'calibrating' l_3 to target values for the model probabilities $p_{1,2}(t) = \mathbb{P}(\tau_1 < t, \tau_2 < t)$ of default of both names up to various time horizons t. More precisely, given a target for the function $p_{1,2}(t)$, one plugs it, together with the functions $p_1(t)$ and $p_2(t)$, into (25), to deduce $L_3(t)$.

Remark 6.2 Regarding the derivation of a target for $p_{1,2}(t)$, note the following relation between $p_{1,2}(t)$ and a standard static Gaussian copula *asset correlation* ρ at the horizon t:

$$p_{1,2}(t) = \mathcal{N}_2^{\rho}\left(\mathcal{N}_1^{-1}(p_1(t)), \mathcal{N}_1^{-1}(p_2(t))\right) , \tag{26}$$

where $\mathcal{N}_1$ denotes the standard Gaussian c.d.f., and $\mathcal{N}_2^{\rho}$ denotes a bivariate centered Gaussian c.d.f. with one-factor Gaussian copula correlation matrix of parameter ρ. A target value for $p_{1,2}(t)$ can thus be obtained by plugging values extracted from the market for ρ, $p_1(t)$ and $p_2(t)$ into the RHS of (26). In particular a 'market' static Gaussian asset correlation ρ can be retrieved from the Basel II correlations per asset class (cf. [3, pages 63 to 66]).

6.3 Calibration

We aim at calibrating the model to marginal CDS curves and to an asset correlation ρ (see Remark 6.2). We assume that the functions f_1 , f_2 and l_3 are piecewise constant functions of time.

We denote by $(T_1, ..., T_m)$ the term structure of the maturities of the market CDS written on the counterparty and on the reference entity, and we set $\Delta_j = T_j - T_{j-1}$, with the convention $T_0 = 0$. One then proceeds in four steps as follows:

• One bootstraps the CDS curve for both names i into a piecewise constant c.d.f. $p_i(\cdot)$, for $i = 1, 2$, yielding

$$p_i(t) = p_i(T_j) \text{ on } T_{j-1} \leq t < T_j ,$$

• Next, given $p_1(t)$, $p_2(t)$ and ρ, one computes $p_{1,2}(t) = \mathbb{P}(\tau_1 < t, \tau_2 < t)$ via (26).

• The relation (25) yields a system of m linear equations in the m unknowns $l_{3,1}, ..., l_{3,m}$.

$$\begin{cases} \Delta_1 l_{3,1} + \cdots + \Delta_j l_{3,j} = -\ln \frac{\mathbb{P}(\tau_1 > T_j, \tau_2 > T_j)}{\mathbb{P}(\tau_1 > T_j)\mathbb{P}(\tau_2 > T_j)} \\ \\ \text{subject to } l_{3,j} \geq 0 , \ j = 1, ..., m \end{cases}$$

• At last, formula (21) results in two systems of m linear equations in the $m + 2$ unknowns $X_0^i, \mu_i, f_{i,1}, ..., f_{i,m}$. That is, for $i = 1, 2$,

$$\begin{cases} \delta\phi(T_j)X_0^i + \delta\xi(T_j)\mu_i + \Delta_1 f_{i,1} + \cdots + \Delta_j f_{i,j} = -\ln \ \mathbb{P}(\tau_i > T_j) \\ \\ \text{subject to } X_0^i \geq 0 , \ \mu_i \geq 0 , \ f_{i,j} \geq l_{3,j}, \ j = 1, ..., m. \end{cases}$$

In practice these equations are solved in the sense of mean-square minimization under constraints.

7 A Variant of the Model with Extended CIR Intensities

In this section, we propose a variant of the general model of section 3, defined in terms of extended CIR factor processes. By comparison with (3), one thus chooses a specific, affine form of the factors, but one also lets the joint defaults intensity l_3 be stochastic, via a 'new' factor X^3. In particular, one models the factors X^is as affine processes of the form

$$dX^i_t = \eta(\mu_i(t) - X^i_t)dt + \nu\sqrt{X^i_t}dW^i_t \ ,$$

with W_1 and W_2 correlated at the level ϱ and W_3 independent from W_1 and W_2. Note in this regard that the factors X^is have the same coefficients but for $\mu_i(t)$, to the effect that $\widetilde{X}^i := X^i + X^3$, for $i = 1, 2$, is again an extended CIR process, with parameters η, $\widetilde{\mu}_i(t) = \mu_i(t) + \mu_3(t)$ and ν.

Let as before $\mathrm{H} = (H^1, H^2)$ and let now $\mathrm{X} = (X^1, X^2, X^3)$.

One thus considers a Markovian model of the pair (X, H) relative to its natural filtration $\mathbb{F}$, with generator of (X, H) given by, for $u = u(t, x, e)$ with $t \in \mathbb{R}_+, x = (x_1, x_2, x_3) \in \mathbb{R}^3, e = (e_1, e_2) \in \{0, 1\}^2$:

$$\mathcal{A}u(t,x,e) = \partial u(t,x,e) + \sum_{1\le i\le 2} l_i(t,x_i)\left(u(t,x,e^i) - u(t,x,e)\right) + l_3(t,x_3)\left(u(t,x,1,1) - u(t,x,e)\right)$$
$$+ \sum_{1\le i\le 3}\Big(\eta(\mu_i(t) - x_i)\partial_{x_i}u(t,x,e) + \frac{1}{2}\nu^2 x_i \partial^2_{x_i^2}u(t,x,e)\Big) + \varrho\nu^2\sqrt{x_1x_2}\partial^2_{x_1,x_2}u(t,x,e) \ ,$$

where, for $i = 1$ to 3:
• the default intensity function l_i is of the form

$$l_i(t, x_i) = x_i + g_i(t) \ , \tag{27}$$

• the coefficients η, ν are non-negative constants and $\mu_i(\cdot)$s are non-negative functions of time.

The $\mathbb{F}$ – intensity matrix-function of H is now given by

$$A(t,x) = \begin{bmatrix} -l(t,x) & l_1(t,x_1) & l_2(t,x_2) & l_3(t,x_3) \\ 0 & -q_2(t,\widetilde{x}_2) & 0 & q_2(t,\widetilde{x}_2) \\ 0 & 0 & -q_1(t,\widetilde{x}_1) & q_1(t,\widetilde{x}_1) \\ 0 & 0 & 0 & 0 \end{bmatrix} ,$$

with, for every $i = 1, 2$,

$$\widetilde{x}_i = x_i + x_3 \quad , \quad q_i(t,\widetilde{x}_i) = l_i(t,x_i) + l_3(t,x_3) = \widetilde{x}_i + g_i(t) + g_3(t)$$

and $l = l_1 + l_2 + l_3$. Under standard regularity and growth assumptions on the coefficients of $\mathcal{A}$, one then has the following variant of Proposition 3.1,

Proposition 7.1 **(i)** *For every $i = 1, 2$, the process $(\widetilde{X}^i, H^i)$ is an $\mathbb{F}$-Markov process, with generator of $(\widetilde{X}^i, H^i)$ given by, for $u_i = u_i(t, \widetilde{x}_i, e_i)$, with $t \in \mathbb{R}_+, \widetilde{x}_i \in \mathbb{R}, e_i \in \{0, 1\}$:*

$$\begin{aligned}\mathcal{A}^i u_i(t, \widetilde{x}_i, e_i) =& \partial_t u_i(t, \widetilde{x}_i, e_i) + \eta(\widetilde{\mu}_i(t) - \widetilde{x}_i)\partial_{\widetilde{x}_i} u_i(t, \widetilde{x}_i, e_i) + \frac{1}{2}\nu^2 x_i \partial^2_{\widetilde{x}_i^2} u_i(t, \widetilde{x}_i, e_i) \\ &+ q_i(t, \widetilde{x}_i)\left(u_i(t, \widetilde{x}_i, 1) - u_i(t, \widetilde{x}_i, e_i)\right) .\end{aligned} \tag{28}$$

The $\mathbb{F}$ – intensity matrix function of H^i is thus given by

$$A_i(t, \widetilde{x}_i) = \begin{bmatrix} -q_i(t, \widetilde{x}_i) & q_i(t, \widetilde{x}_i) \\ 0 & 0 \end{bmatrix}$$

In other words, the process M^i defined by, for $i = 1, 2$,

$$M^i_t = H^i_t - \int_0^t (1 - H^i_s) q_i(s, \widetilde{X}^i_s) ds ,$$

is an $\mathbb{F}$-martingale.
(ii) *One has, for every $t \geq 0$,*

$$\mathbb{P}(\tau_i > t) = \mathbb{E}\exp\big(-\int_0^t q_i(u, \widetilde{X}^i_u) du\big) \quad , \quad \mathbb{P}(\tau_1 \wedge \tau_2 > t) = \mathbb{E}\exp\big(-\int_0^t l(u, \mathrm{X}_u) du\big) \tag{29}$$

One thus gets in the terminology of [5] *a Markovian copula model* (X, H) *with marginals* $(\widetilde{X}^i, H^i)$, for $i = 1, 2$ — or, in the 'common factor case' $X^1 = X^2 = X$, with marginals $(\widetilde{X}, H^i)$, where we set $\widetilde{X} = X + X^3$.

Let, for $\widetilde{x}_1 \in \mathbb{R}_+$,

$$p(t, \widetilde{x}_1) = (1 - R_1) q_1(t, \widetilde{x}_1) - \kappa .$$

One then has much like in Proposition 4.3(i) (analogs of Propositions 4.3(ii) and (iii) could be derived as well if wished),

Proposition 7.2 *The price of the risk-free CDS admits the representation:*

$$P_t = (1 - H^1_t) v(t, \widetilde{X}^1_t) ,$$

for a pre-default pricing function $v = v(t, \widetilde{x}_1)$ solving the following pricing PDE:

$$\begin{cases} v(T, \widetilde{x}_1) = 0 , \quad \widetilde{x}_1 \in \mathbb{R} \\ \Big(\partial_t + \eta(\widetilde{\mu}_1(t) - \widetilde{x}_1)\partial_{\widetilde{x}_1} + \frac{1}{2}\nu^2 \widetilde{x}_1 \partial^2_{\widetilde{x}_1^2}\Big) v(t, \widetilde{x}_1) - \big(r + q_1(t, \widetilde{x}_1)\big) v(t, \widetilde{x}_1) + p(t, \widetilde{x}_1) = 0 , \\ \qquad t \in [0, T), \, \widetilde{x}_1 \in \mathbb{R} ; \end{cases}$$

Hedging could also be discussed analogously as in Section 5 for (2F).

7.1 Implementation

Let us consider the parametrization stated in (27), with $g_i = 0$, therein. We assume that the $\mu_i(\cdot)$s are piecewise constant functions,

$$\mu_i(t) = \mu_{i,j} \ , \ \text{for } t \in [T_{j-1}, T_j).$$

The marginal intensity processes $q_i(t, \widetilde{X}^i_t)$s are then extended CIR processes (cf. (28)) with the following piecewise constant 'long-term mean' function $\widetilde{\mu}_i(\cdot)$,

$$\widetilde{\mu}_i(t) = \mu_{i,j} + \mu_{3,j} \ , \ \text{for } t \in [T_{j-1}, T_j) \ .$$

We will refer to this model parametrization as (3F). Under this specification, one has the following proposition for the pricing function of a risk-free CDS on the firm. Let the functions $\widetilde{\mathcal{D}}_1$ and $\widetilde{\mathcal{E}}_1$ be defined as in Proposition A.2, with $\mu(\cdot) = \widetilde{\mu}_1(\cdot)$ therein.

Proposition 7.3 *Assuming (3F), one has,*

$$\beta(t)v(t, \widetilde{x}_1) = \int_t^T \beta(s)\big((1 - R_1)\widetilde{\mathcal{D}}(s, t, \widetilde{x}_1) - \kappa\big)\widetilde{\mathcal{E}}(s, t, \widetilde{x}_1, 0)ds.$$

Proof. Recall from Proposition 7.2(i) that

$$v(t, \widetilde{x}_1) = \mathbb{E}\Big(\int_t^T e^{-\int_t^s (r + q_1(\zeta, \widetilde{X}^1_\zeta))d\zeta} p(s, \widetilde{X}^1_s)ds \,\Big|\, \widetilde{X}^1_t = \widetilde{x}_1\Big) \ ,$$

with

$$p(s, \widetilde{X}^1_s) = (1 - R_1)\widetilde{X}^1_s - \kappa.$$

The result thus follows by an application of Proposition A.2. □

Also, the spread at time 0 of a risk-free CDS of maturity T on the firm, is given by

$$\kappa_0(T) = (1 - R_1)\frac{\int_0^T \beta(s)\widetilde{\mathcal{D}}_1(s, 0, X^1_0)\widetilde{\mathcal{E}}_1(s, 0, X^1_0, 0)ds}{\int_0^T \beta(s)\widetilde{\mathcal{E}}_1(s, 0, X^1_0, 0)ds} \ .$$

As for the forward spread, the counterparty of formulae (22) for this variant of the model, is

$$\kappa_0(T_a, T_b) = (1 - R_1)\frac{\int_{T_a}^{T_b} e^{-rs}\widetilde{\mathcal{D}}_\delta(s, 0, X^1_0)\widetilde{\mathcal{E}}_\delta(s, 0, X^1_0)ds}{\int_{T_a}^{T_b} e^{-rs}\widetilde{\mathcal{E}}_\delta(s, 0, X^1_0)ds} \ . \tag{30}$$

Also, the price at time 0 of CDS options with strike κ are given by

$$e^{-rT_a}\mathbb{E}\big[(1 - H^1_{T_a})v(T_a, \widetilde{X}^1_{T_a})^+\big] \text{ for a payer CDS option ,} \tag{31}$$

$$e^{-rT_a}\mathbb{E}\big[(1 - H^1_{T_a})v(T_a, \widetilde{X}^1_{T_a})^-\big] \text{ for a receiver CDS option ,} \tag{32}$$

where $v(t, \widetilde{x}_1)$ is the price function given by Proposition 7.3.

As in the previous case, the input to the calibration is an asset correlation ρ and the piecewise constant marginal cumulative default probabilities obtained by bootstrapping from the related CDS curves. For simplicity of calibration, the volatility parameter ν and the mean-reversion η are assumed to be given (as opposed to calibrated), whereas for each factor X^i the initial value X_0^i and $\mu_{i,1}, ..., \mu_{i,m}$ are calibrated.

Using identities (29) and Corollary A.3, the following expressions follows for the marginal and joint survival probabilities:

$$\mathbb{P}(\tau_i > T_j) = \mathbb{E}\Big(\exp\big(-\int_0^{T_j} \widetilde{X}_s^i ds\big)\Big) = \exp\big(-a_{j,0}\widetilde{X}_0^i - \sum_{k=1}^{j} \widetilde{\mu}_{i,k}\xi(\Delta_k, a_{j,k})\big)$$

and

$$\begin{aligned}
\mathbb{P}(\tau_1 > T_j, \tau_2 > T_j) &= \mathbb{E}\Big(\exp\big(-\int_0^{T_j} (X_s^1 + X_s^2 + X_s^3) ds\big)\Big) \\
&= \mathbb{P}(\tau_1 > T_j)\mathbb{P}(\tau_2 > T_j)\exp\big(a_{j,0}X_0^3 + \sum_{k=1}^{j} \mu_{3,k}\xi(\Delta_k, a_{j,k})\big).
\end{aligned}$$

where the coefficients $a_{j,k}$ are given in (38).

One can then follow the same lines as in Section 6.3, to obtain the following three systems of linear equations with constraints. Each system consists of m equations in $m+1$ unknowns: For $j = 1, ..., m$,

$$\begin{cases}
a_{j,0}X_0^3 + \sum_{k=1}^{j} \xi(\Delta_k, a_{j,k})\mu_{3,k} = \ln \frac{\mathbb{P}(\tau_1 > T_j, \tau_2 > T_j)}{\mathbb{P}(\tau_1 > T_j)\ \mathbb{P}(\tau_2 > T_j)} \\
\\
\text{subject to } \ X_0^3 \geq 0\ ,\ \ \mu_{3,k} \geq 0\ ,\ \ k = 1, ..., m.
\end{cases}$$

For $i = 1, 2$,

$$\begin{cases}
a_{j,0}\widetilde{X}_0^i + \sum_{k=1}^{j} \xi(\Delta_k, a_{j,k})\widetilde{\mu}_{i,k} = -\ln\ \mathbb{P}(\tau_i > T_j) \\
\\
\text{subject to } \ \widetilde{X}_0^i \geq X_0^3\ ,\ \ \widetilde{\mu}_{i,k} \geq \mu_{3,k}\ ,\ \ k = 1, ..., m.
\end{cases}$$

In practice these equations are solved in the sense of mean-square minimization under constraints.

8 Numerical Results

Our aim is to assess by means of numerical experiments the impact on the counterparty risk exposure of:
• ρ, a constant asset correlation between the firm and the counterparty,
• p_2, the cumulative distribution function of the default time τ_2 of the counterparty,
• ν, the volatility of the factors.

The numerical tests below have been done using the following model parameterizations:

(0F) No stochastic factor, as in (20) with $\delta = 0$,

(2F) Two independent CIR++ factors, as in (20) with $\delta = 1$,

(3F) Three independent extended CIR factors as of subsection 7.1 with $\varrho = 0$.

The mean-reversion parameter η is fixed to 10%. The recovery rates are set to 40% and the risk-free rate r is taken equal to 5%.

8.1 Calibration to Market Data

In the following example, we consider four CDSs written on the reference name UBS AG, with different counterparties: Gaz de France, Carrefour, AXA and Telecom Italia SpA, referred to in the sequel as CP1, CP2, CP3 and CP4. For each counterparty we consider six CDSs with maturities of one, two, three, five, seven and ten years, corresponding to data of March 30, 2008. Table 1 includes market CDS spreads on the five names in consideration and for the six different maturities. The bootstrapped piecewise constant c.d.f. of the five names are represented in Table 2. The counterparties are ordered form the less risky one, CP1, to the most risky one, CP4.

		1 year	2 years	3 years	5 years	7 years	10 years
Ref	UBS AG	90	109	129	147	148	146
CP1	Gaz de France	27	35	42	53	57	61
CP2	Carrefour	34	42	53	67	71	76
CP3	AXA	72	83	105	128	129	128
CP4	Telecom Italia SpA	99	157	210	243	255	262

Table 1: Market spreads in bps for different time horizons on March 30, 2008.

	1 year	2 years	3 years	5 years	7 years	10 years
Ref	.0146	.0355	.0631	.1185	.1612	.2193
CP1	.0044	.0116	.0212	.0445	.0664	.1005
CP2	.0056	.0138	.0264	.0558	.0822	.1246
CP3	.0118	.0269	.0517	.1042	.1434	.1964
CP4	.0155	.0504	.1026	.1903	.2662	.3670

Table 2: Default probabilities for different maturities.

Tables 3 and 4 represent the calibration error in basis points of (2F) and (3F), respectively. Precisely, in each table, we consider

$$\mathrm{er}_i(t) = 10^4 \times \frac{|p_i(t) - \widehat{p}_i(t)|}{p_i(t)} \ , \ \mathrm{er}_{1,2}(t) = 10^4 \times \frac{|p_{1,2}(t) - \widehat{p}_{1,2}(t)|}{p_{1,2}(t)}$$

where $\widehat{p}_1, \widehat{p}_2, \widehat{p}_{1,2}$ are obtained from equations (7) and (29) using the calibrated parameters. The corresponding errors in the case of (0F) are 0.0000 bp. The difference between market

		Maturities							
		1 year	2 years	3 years	5 years	7 years	10 years	mean	max
CP1	er_1	0.2500	0.9630	0.8560	1.1390	2.1600	1.7600	1.1880	2.1600
	er_2	0.0191	0.0112	0.3790	0.0612	0.4027	0.0198	0.1488	0.4027
	$er_{1,2}$	0.1790	0.7700	0.5350	1.2040	1.1120	0.7610	0.7600	1.2040
CP2	er_1	1.9240	1.8000	1.0500	1.3020	0.2590	2.1340	1.4110	2.1340
	er_2	0.3817	0.0726	0.0714	0.7904	0.3275	0.3992	0.3404	0.7904
	$er_{1,2}$	0.2471	0.1127	0.0644	0.0442	0.0990	0.0213	0.0981	0.2471
CP3	er_1	0.4150	1.2380	0.4110	0.3430	1.3280	0.3410	0.6790	1.3280
	er_2	0.4730	0.8080	0.5390	0.6290	3.9800	0.8040	1.2050	3.9800
	$er_{1,2}$	0.0494	0.0437	0.0065	0.0011	0.0116	0.0029	0.0192	0.0494
CP4	er_1	0.0364	0.1808	0.2382	0.1823	0.0988	0.3502	0.1811	0.3502
	er_2	0.0586	0.0167	0.0095	0.0376	0.0206	0.0285	0.0286	0.0586
	$er_{1,2}$	0.2097	0.3922	0.2680	0.0917	0.0815	0.1537	0.1994	0.3922

Table 3: Relative error in bps of the cumulative probabilities p_1, p_2 and $p_{1,2}$ in the case (2F) with $\rho = 40\%$.

spreads and calibrated model spreads are represented in Tables 5, 6 and 7, respectively.

8.2 CVA Stylized Features

Figure 1 shows the Credit Valuation Adjustment at time 0 of a risky CDS on the reference name UBS AG, as a function of the volatility parameter ν of the CIR factors X^is.

The graphs on the left of this figure show the results obtained from the parametrization (2F) while the graphs on the right correspond to the case of (3F). On each graph the asset correlation ρ is fixed, with from top to down $\rho = 5\%$, 10%, 40% and 70%. The four curves on each graph of Figure 1 correspond to Θ_0 calculated for a risky CDS of maturity $T = 10$ years between Ref and CP1, CP2, CP3, CP4, respectively.

On this data set we observe that Θ_0 is:
• increasing in the default risk of the counterparty,
• increasing in the asset correlation ρ,
• slowly increasing in the volatility ν of the common factor.

In Table 8, one can see the values of Θ_0 calculated within the parametrization (0F), that is with no stochastic factor. Note that for a CDS written on Ref, the risk-free value of the default leg is equal to $\mathrm{DL}_0 = 0.1031$.

For the receiver case, $\bar{\Theta}_0$ on the reference name UBS AG is shown in Figure 2 as a function of the volatility parameter ν, for both parameterizations (2F) and (3F). Note that $\bar{\Theta}_0$ is much smaller (for a common unit nominal), and more dependent on ν, than Θ_0. This is mostly due to the absence of the common jump term in the $\overline{\mathrm{CVA}}$ (see Remark 2.2). Also, $\bar{\Theta}_0$ is *decreasing* in the asset correlation ρ.

To give an idea about the **execution times** of the three model parameterizations, it is worth mentioning that, in these experiments:
• a calibration takes about 0.01, 0.30 and 0.35 seconds for the case of (0F), (2F) and (3F),

		Maturities							
		1 year	2 years	3 years	5 years	7 years	10 years	mean	max
CP1	er_1	5.0422	14.190	2.1420	15.050	15.920	0.4151	8.7940	15.920
	er_2	0.7888	1.6940	2.4450	0.0024	2.7510	1.0728	1.4591	2.7515
	$er_{1,2}$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
CP2	er_1	0.5700	2.1950	2.6910	1.4200	0.4247	0.0426	1.2240	2.6910
	er_2	0.3025	0.1581	0.4489	0.4358	0.0444	0.4780	0.3113	0.4780
	$er_{1,2}$	0.0000	0.0000	0.0000	0.0006	0.0006	0.0002	0.0002	0.0006
CP3	er_1	0.0434	0.3486	0.4462	0.1976	0.1464	0.0282	0.2018	0.4462
	er_2	25.809	82.406	87.861	34.909	68.141	0.0325	49.860	87.861
	$er_{1,2}$	0.0001	0.0012	0.0020	0.0036	0.0063	0.0054	0.0032	0.0063
CP4	er_1	1.5396	5.3363	24.188	59.728	40.733	0.9771	22.084	59.728
	er_2	0.0652	3.8962	4.6377	4.6433	3.8499	0.9716	3.0106	4.6432
	$er_{1,2}$	0.0002	0.0030	0.0102	0.0176	0.0263	0.0081	0.0109	0.0263

Table 4: Relative error in bps of the cumulative probabilities p_1, p_2 and $p_{1,2}$ in the case (3F) with $\rho = 40\%$.

	1 year	2 years	3 years	5 years	7 years	10 years	mean	max
Ref	0.2920	0.1622	0.3957	0.3296	0.2537	0.1964	0.2716	0.3957
CP1	0.1285	0.0693	0.1556	0.1080	0.0829	0.0639	0.1014	0.1556
CP2	0.1067	0.0576	0.1897	0.1370	0.1054	0.0819	0.1130	0.1897
CP3	0.0096	0.0052	0.3108	0.2665	0.2052	0.1586	0.1593	0.3108
CP4	0.0098	0.0060	0.4711	0.5125	0.4097	0.3310	0.2900	0.5125

Table 5: bp-Differences between market spreads and calibrated spreads in the case of (0F) with $\rho = 40\%$.

	1 year	2 years	3 years	5 years	7 years	10 years	mean	max
Ref	0.3343	0.2167	0.4315	0.3980	0.3120	0.3489	0.3402	0.4315
CP1	0.0719	0.0781	0.0131	0.0150	0.0305	0.1040	0.0521	0.1040
CP2	0.0345	0.0028	0.1352	0.0852	0.0598	0.0833	0.0668	0.1352
CP3	0.0203	0.0088	0.2876	0.2426	0.1461	0.1855	0.1485	0.2876
CP4	0.0698	0.0537	0.5219	0.5614	0.4584	0.3976	0.3438	0.5614

Table 6: bp-Differences between market spreads and calibrated spreads in the case of (2F) with $\rho = 40\%$.

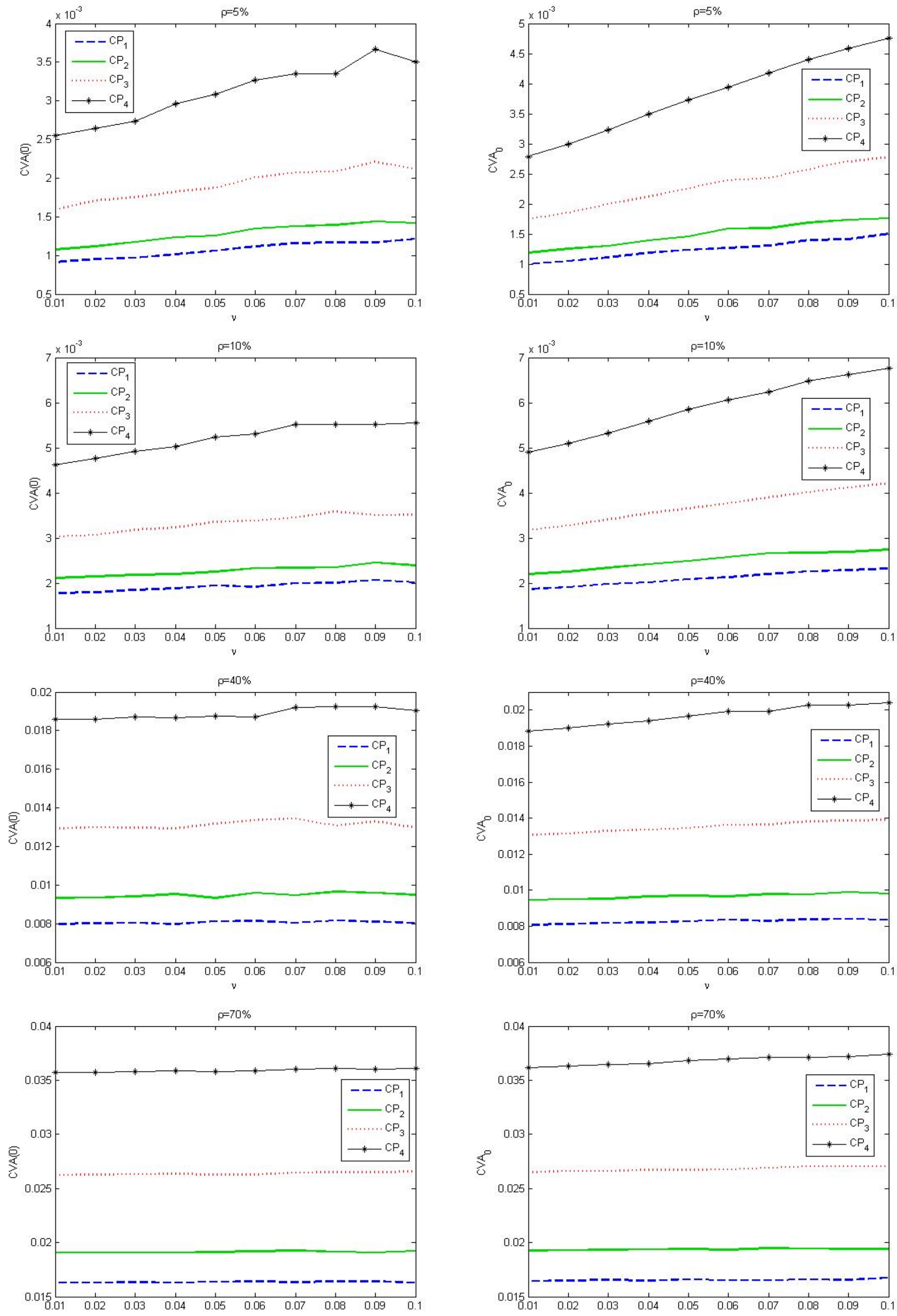

Figure 1: Θ_0 versus ν for the payer CDS on Ref. The graphs on the left column correspond to the case (2F) and those of the right column correspond to (3F). In each graph ρ is fixed. From top to down $\rho = 5\%$, $\rho = 10\%$, $\rho = 40\%$ and $\rho = 70\%$.

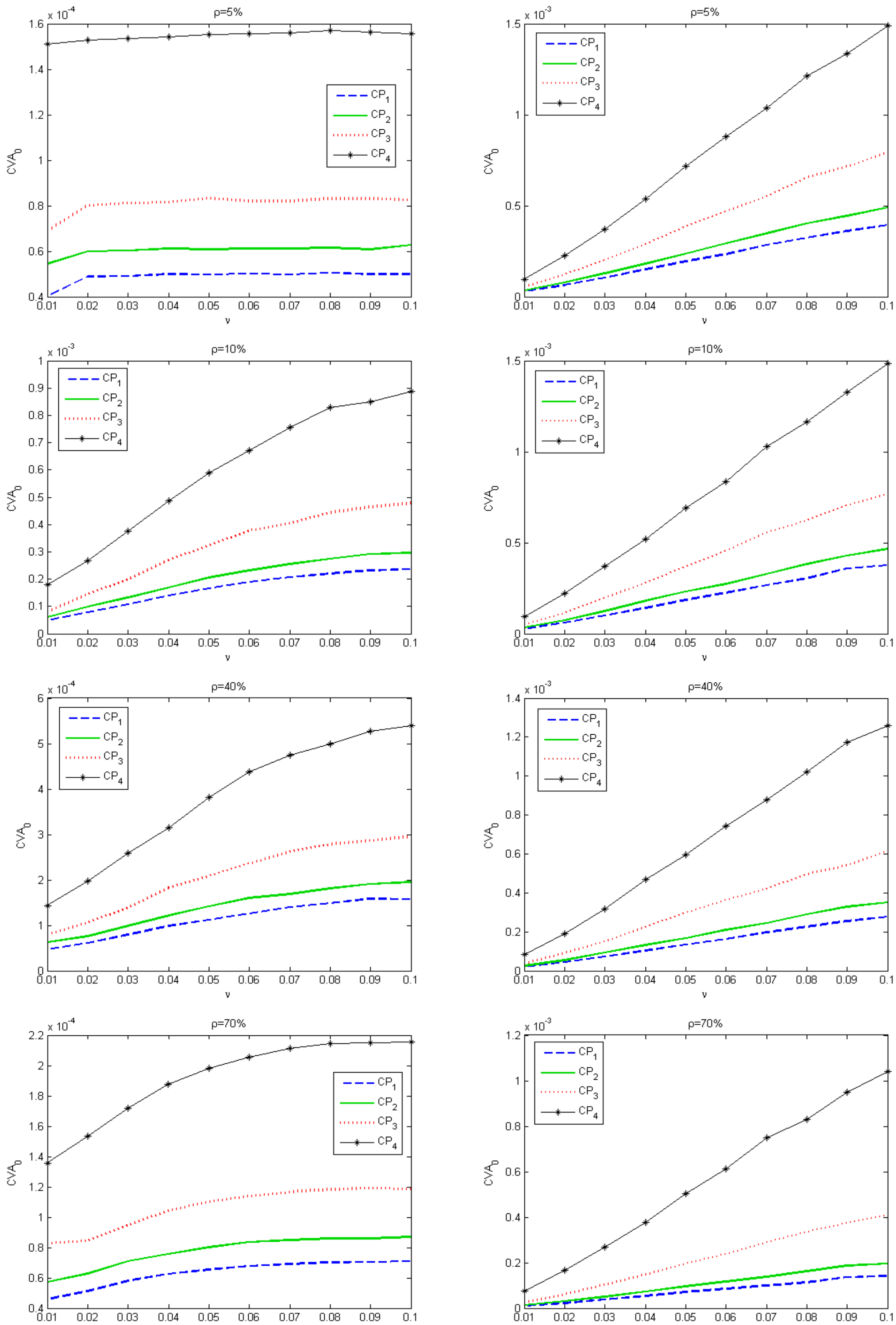

Figure 2: $\bar{\Theta}_0$ versus ν for the receiver CDS on Ref. The graphs on the left column correspond to the case (2F) and those of the right column correspond to (3F). In each graph ρ is fixed. From top to down $\rho = 5\%$, $\rho = 10\%$, $\rho = 40\%$ and $\rho = 70\%$.

	1 year	2 years	3 years	5 years	7 years	10 years	mean	max
Ref	1.8110	1.6440	0.6820	0.8820	0.4950	0.4790	0.9988	1.8110
CP1	0.7730	0.6560	0.4300	0.2370	0.2130	0.0750	0.3973	0.7730
CP2	0.7400	0.8190	0.4300	0.2030	0.2140	0.1250	0.4218	0.8190
CP3	1.1690	0.9320	1.4230	0.5160	0.6940	0.4710	0.8675	1.4230
CP4	5.6840	3.5300	1.7190	0.6740	0.5720	0.4910	2.1117	5.6840

Table 7: bp-Differences between market spreads and calibrated spreads in the case of (3F) with $\rho = 40\%$.

	$\rho = 5\%$	$\rho = 10\%$	$\rho = 40\%$	$\rho = 70\%$
CP1	.0009	.0018	.0080	.0163
CP2	.0011	.0021	.0093	.0190
CP3	.0016	.0030	.0129	.0262
CP4	.0025	.0047	.0186	.0358

Table 8: Θ_0 for CDSs written on Ref in the case (0F).

respectively;
• a computation of CVA takes about 0.015, 5.0 and 12 seconds for the cases of (0F), (2F) and (3F), respectively.

8.3 Case of a Low-Risk Reference Entity

In the previous example, except in the low ρ cases, the dependency of Θ_0 on ν was rather limited (see Figure 1). For a low-risk reference entity, however, ν is expected to have more impact on Θ_0, including for larger ρ's. To assess this numerically we thus now consider a low-risk obligor, referred to as Ref', whose piecewise constant c.d.f. is given in Table 9. For a CDS written on Ref', the risk-free value of the default leg is equal to $\mathrm{DL}'_0 = 0.0240$. On each graph of Figure 3, the asset correlation is fixed to $\rho = 5\%, 10\%, 40\%$ or 70%.

1 year	2 years	3 years	5 years	7 years	10 years
.0100	.01500	.0200	.0300	.0400	.0500

Table 9: Default probabilities of Ref'

One can see that Θ_0 is significantly sensitive to ν, and even extremely so in the case of low correlations ρ. For comparison Table 10 shows the values of Θ_0 calculated within the parametrization (0F).

	$\rho = 5\%$	$\rho = 10\%$	$\rho = 40\%$	$\rho = 70\%$
CP1	.0002	.0006	.0031	.0073
CP2	.0003	.0007	.0035	.0080
CP3	.0004	.0009	.0046	.0096
CP4	.0007	.0014	.0061	.0108

Table 10: Θ_0 for CDSs written on Ref' in the case (0F).

8.4 Spread Options Implied Volatilities

The goal of this subsection is to assess the level of spreads' volatility implied by the alternative model parameterizations. To this end we will compute the implied volatility of payer and receiver CDS options written on individual names.

The explicit Black formula for the price of a CDS option can be found in Brigo [9] (see for example Eq. (28) therein). The Markov copula model prices are given by formulae (23) and (24) for the case of (0F) and (2F) and by formulae (31) and (32) for the case of (3F).

For numerical tests we consider CDSs of 7 year maturity on credit names CP1, Ref and CP4 (cf. Table 1). Also, we consider payer and receiver CDS options with maturity of 3 years on these three names and with strike $K = 65$, 150 and 250 bp, respectively. So, with notation introduced for forward spread, $T_a = 3$ and $T_b = 10$.

Figure 4 shows the implied volatility versus ν of receiver CDS options written on the three names and for the parameterizations (2F) and (3F), the case of (0F) corresponding essentially to ν tending to 0 in (2F) or (3F) (in particular the implied volatility of all the three receiver CDS options is then equal to zero).

The curves in the case (2F) are represented on the left column and those of (3F) are on the right column. The graphs on the top, middle and bottom of this figure correspond, respectively, to receiver CDS option written on names CP1, Ref and CP4. Recall from Table 1 that CP1, Ref and CP4 are, respectively, low-risk, middle-risk and high-risk names, in the sense of their 10-years CDS spread. On Figure 5 the same graphs for payer CDS options are represented.

One observes that for the same level of ν, implied volatility in the case of (3F) is typically much higher than that of (2F), which was expected since the joint defaults intensity l_3 is deterministic in (2F), whereas intensities are 'fully stochastic' in (3F). Also, for a fixed level of ν, the implied volatility is decreasing in riskiness of the credit name. In other words, to achieve a given implied volatility, the riskier the credit name, the lower the implied ν.

For the parametrization (3F), the implied volatility curves are non-decreasing in ν, in both payer and receiver cases. As is observed, this is not the case for the parametrization (2F).

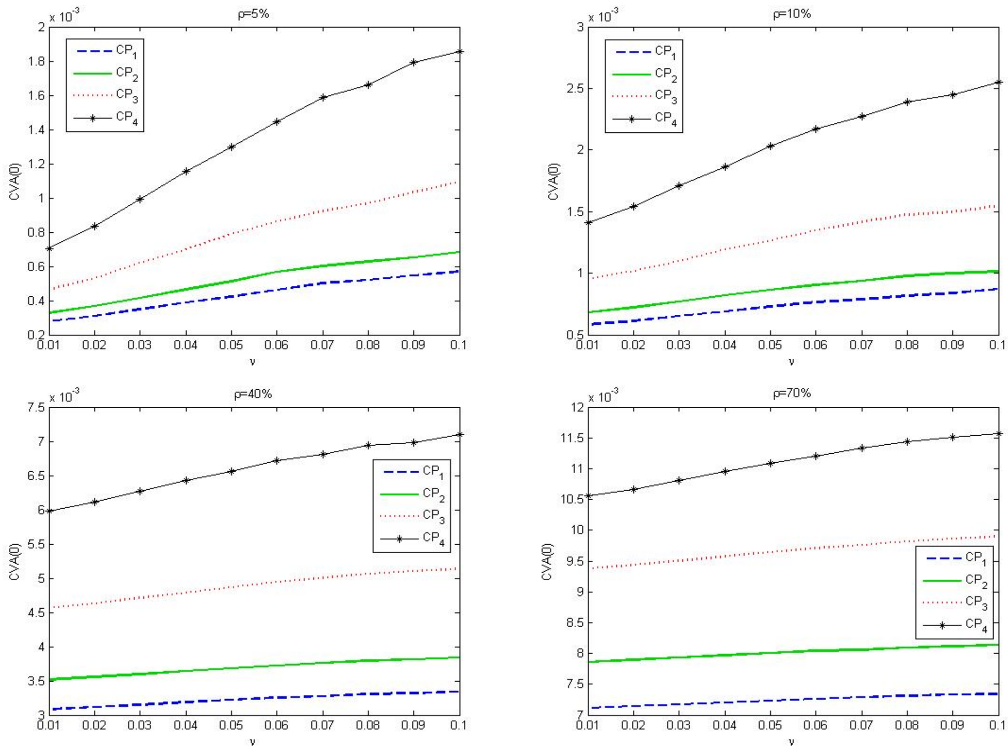

Figure 3: Θ_0 versus ν for a CDS written on Ref' in the case (2F). In each graph ρ is fixed.

8.5 The Analysis of the Contribution of the Joint Default

In this subsection, drawing on the same data set as before, we make a numerical test to assess the contribution of the joint default to the payer credit valuation adjustment (parameter α below), and its contribution to the default scenarios of the counterparty (parameter $\widetilde{\alpha}$ below).

Recall that in our model, Θ_0 has two components, one of which is due to the simultaneous default of the reference entity and the counterparty. Introducing the notation

$$\widehat{\xi}_{(\tau_2)} := (1 - R_1)(1 - R_2)\,\mathbb{1}_{\tau_2=\tau_1<T}\ , \quad \beta(t)\widehat{\Theta}_t = \mathbb{1}_{\{t<\tau_2\}}\mathbb{E}_t\big[\beta(\tau_2)\widehat{\xi}_{(\tau_2)}\big]\ ,$$

we define the parameter α which measures the contribution of the joint default to Θ_0,

$$\alpha = \frac{\widehat{\Theta}_0}{\Theta_0}\ .$$

Inspired by Remark 2.2 (Regime 1), it is interesting to investigate the behavior of the parameter α with respect to the asset correlation ρ, in the case where both $\mathbb{P}(\tau_2$ is small) and $\mathbb{P}(\tau_2 < \tau_1)$ are large. Figure 6, represents α as a function of ρ, for the most risky counterparty CP4, and for the reference entity Ref (on the left) and Ref' (on the right). The three curves on each figure, correspond to the cases (0F), (2F) and (3F). We observe

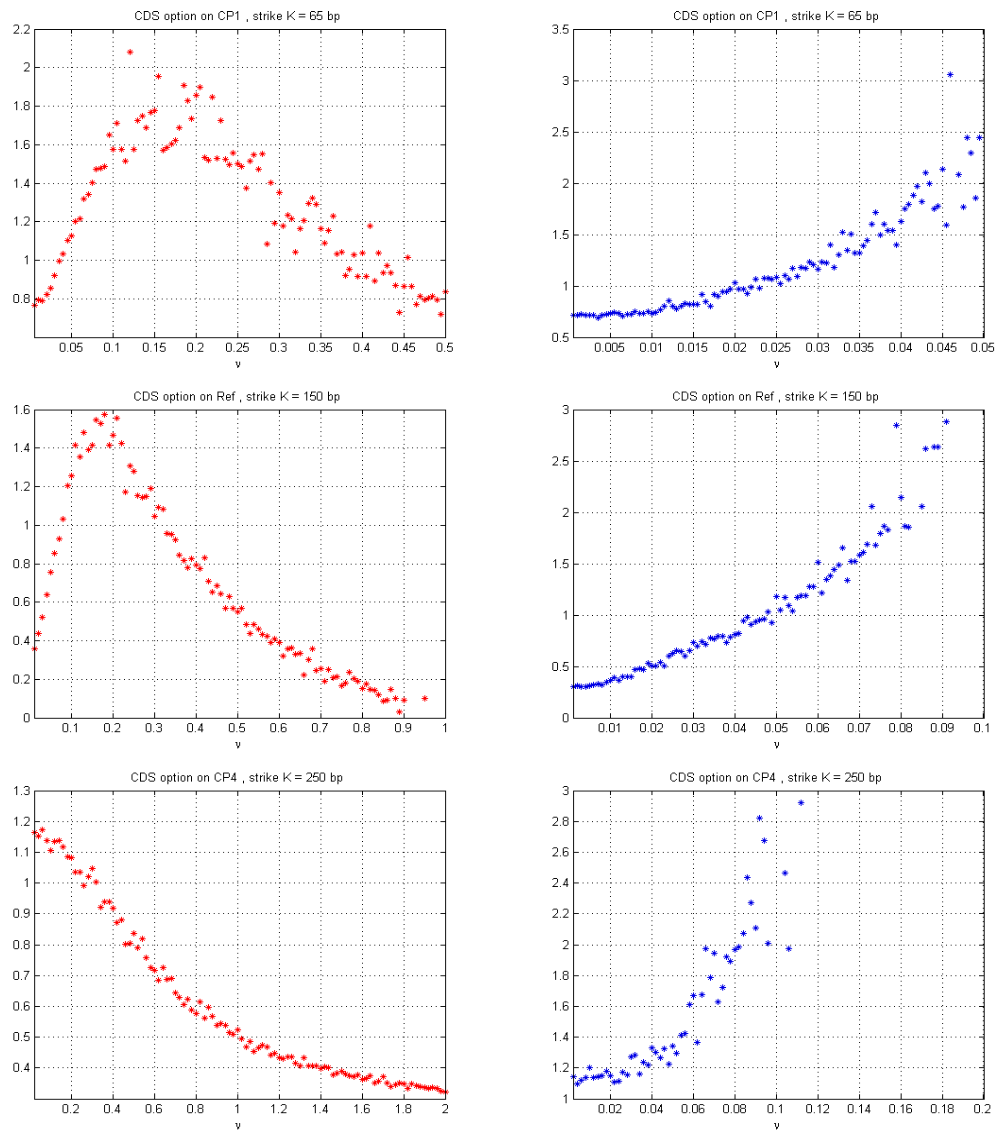

Figure 4: Implied volatility versus ν of payer CDS option written on individual names. The graphs on the left column correspond to the case (2F) and those of the right column correspond to (3F). The graphs on the top, middle and bottom correspond to payer CDS options written on names CP1, Ref and CP4 (with strike K = 65bp, 150bp and 250bp), respectively.

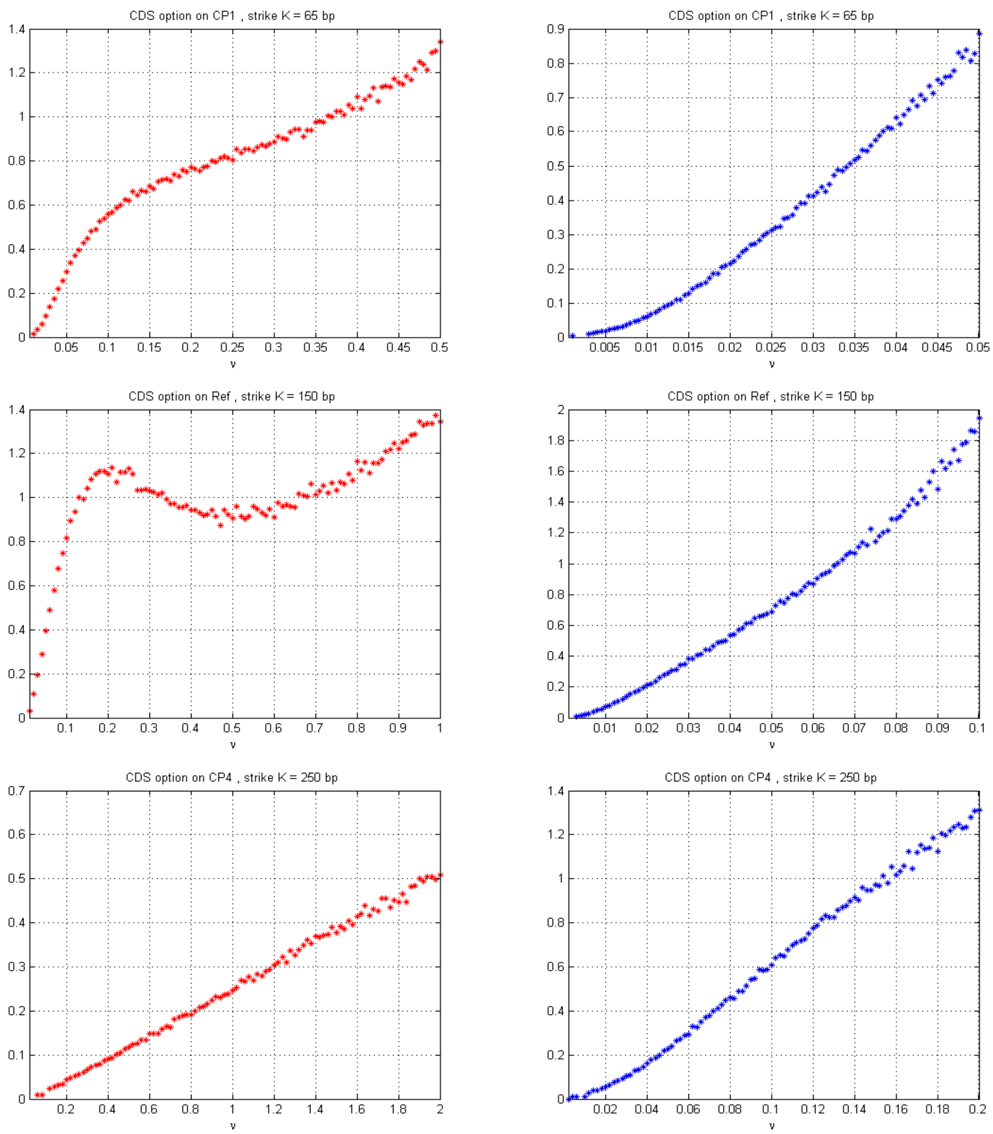

Figure 5: Implied volatility versus ν of receiver CDS option written on individual names. The graphs on the left column correspond to the case (2F) and those of the right column correspond to (3F). The graphs on the top, middle and bottom correspond to receiver CDS options written on names CP1, Ref and CP4 (with strike K = 65bp, 150bp and 250bp), respectively.

that, even in the case of “Regime 1”, and in all the three model parameterizations, α is increasing in ρ and that for $\rho = 1$, one has $\alpha \approx 1$. This means that, in our model, it is the joint default that plays the essential role in the credit valuation adjustment, Θ_0.

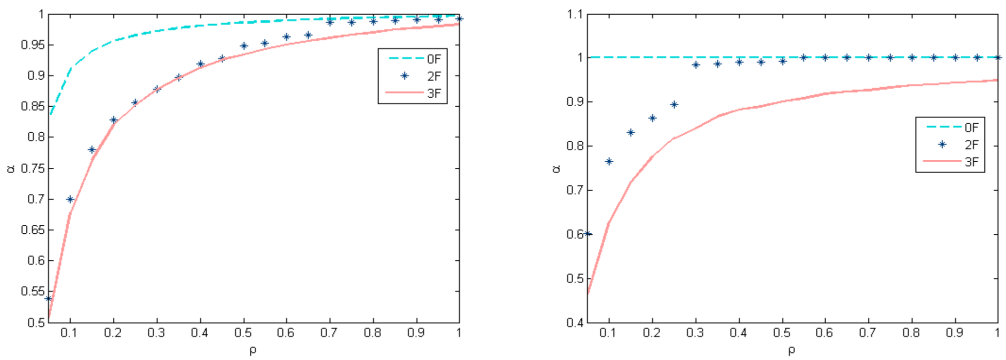

Figure 6: α as a function of ρ for the counterparty CP4, in the case of deterministic intensities (0F), and stochastic intensities (2F) and (3F), both with $\nu = 0.1$. The left graphs correspond to the reference name Ref, and the right graphs correspond to Ref’.

As we have already observed, for a payer CDS, the volatility parameter has a more important impact on Θ_0, in the case of a less risky reference name, and of a lower correlation between the counterparty and the reference name (cf. Figures 1 and 3). This motivates us to define another parameter, $\widetilde{\alpha}$, which represents the proportion of default scenarios of the counterparty due to its simultaneous default with the reference entity

$$\widetilde{\alpha} := \frac{\sharp\{\tau_2 = \tau_1 < T\}}{\sharp\{\tau_2 = \tau_1 < T\} + \sharp\{\tau_2 < \tau_1 \wedge T\}} \approx \frac{\mathbb{P}(\tau_2 = \tau_1 < T)}{\mathbb{P}(\tau_2 = \tau_1 < T) + \mathbb{P}(\tau_2 < \tau_1 \wedge T)} .$$

Thus, in the following numerical experiences, we are interested in the behavior of $\widetilde{\alpha}$ as a function of the asset correlation ρ, for the less risky reference name, Ref’. Table 11 shows $\widetilde{\alpha}$ in the case of deterministic intensities (0F), for the reference name Ref’, different counterparties and different levels of the correlation ρ. It is seen that for all counterparties, $\widetilde{\alpha}$ is increasing in the correlation.

	$\rho = 5\%$	$\rho = 10\%$	$\rho = 40\%$	$\rho = 70\%$
CP1	.0105	.0220	.1160	.2636
CP2	.0099	.0208	.1062	.2333
CP3	.0087	.0180	.0857	.1725
CP4	.0070	.0141	.0596	.1023

Table 11: $\widetilde{\alpha}$ as a function of ρ for Ref’, and the parametrization (0F).

The curves in Figure 7 represent $\widetilde{\alpha}$ as a function of the asset correlation ρ, for the less risky reference entity, Ref’, and four different counterparties. The graph on the left corresponds to (2F) and the graph on the right corresponds to (3F), with the volatility parameter ν fixed and equal to 0.1 for both parameterizations. It is observed that $\widetilde{\alpha}$ is increasing in ρ. This is inline with the our previous observation: For the less risky reference entity and lower level of the correlation ρ, there are more scenarios where the counterparty defaults prior to the reference entity, and hence, the impact on CVA of the volatility parameter, ν, is more important.

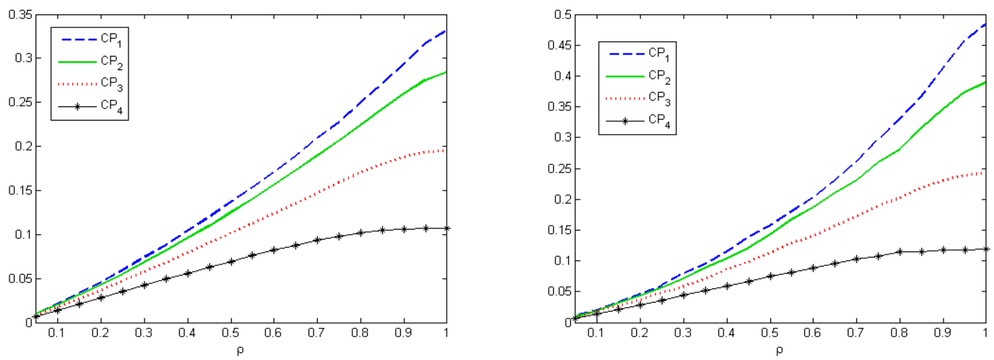

Figure 7: $\widetilde{\alpha}$ as a function of ρ for Ref'. The graph on the left corresponds to (2F), and that on the right corresponds to (3F). In each graph $\nu = 0.1$.

9 Conclusions

One develops in this paper a Markovian model of counterparty credit risk on a CDS. The issue of 'wrong-way risk', which is particularly important in the case of a payer CDS, is represented in the model by the possibility of simultaneous defaults of the counterparty and of the reference firm of the CDS. Since this is a dynamic model of counterparty credit risk, prices and CVAs can be connected to dynamic hedging arguments, as illustrated by our study of mean-variance hedging the CVA of the CDS on the firm by a rolling CDS on the counterparty.

Moreover, we devise, implement and discuss three model specifications.

Our numeric results show that in the case of a payer CDS on a 'risky enough' reference entity and for a sufficient level of correlation between the counterparty and the reference entity, a time-deterministic specification of intensities does a good and quick job in estimating the CVA.

In case of a receiver CDS, or of a payer CDS with low risk reference entity or low level of correlation between the counterparty and the reference entity, the time-deterministic specification of intensities 'misses' a non-negligible component of CVA, due to spreads' volatility. In this case, a stochastic specification of the intensities is preferred (cf. Figures 2, 3 and 1), like a CIR++ specification of the intensities with marginal default intensities given as sums of affine processes and deterministic functions of time.

In this specification the joint defaults intensity of the counterparty and the reference firm is time-deterministic, so that one might wonder whether a fully stochastic specification of the intensities would lead to even higher (possibly more realistic) CVAs. This led us to investigate a third specification of the intensities in the form of extended CIR processes with time-dependent parameters, and no deterministic component anymore. In case of a payer CDS the levels of CVA happen to be quite similar to those got through the CIR++ specification, however for the receiver CDS they may be much larger (as is seen in Figure 2).

A Appendix

Let X be an extended CIR process with dynamics

$$dX_t = \eta(\mu(t) - X_t)dt + \nu\sqrt{X_t}dW_t \tag{33}$$

where η and ν are positive constants and $\mu(\cdot)$ is a non-negative deterministic function.

The following lemma is a standard result in the affine processes literature (see e.g. [17]). Notice that (35) is obtained from (34) with $y = 0$, by differentiating with respect to t.

Lemma A.1 *Consider the process X in (33). If $\mu(\cdot)$ is constant on $[t_0, t]$, then for every $y \geq 0$,*

$$\mathbb{E}\left(e^{-\int_{t_0}^t X_s ds - yX_t} \big| X_{t_0}\right) = e^{-\phi(t-t_0,y)X_{t_0} - \xi(t-t_0,y)\mu} , \tag{34}$$

$$\mathbb{E}\left(X_t e^{-\int_{t_0}^t X_s ds} \big| X_{t_0}\right) = \left(\dot\phi(t-t_0,0)X_{t_0} + \dot\xi(t-t_0,0)\mu\right)e^{-\phi(t-t_0,0)X_{t_0} - \xi(t-t_0,0)\mu} , \tag{35}$$

where ϕ and ξ satisfy the following system of ODE:

$$\begin{cases} \dot\phi(s,y) = -\eta\phi(s,y) - \frac{\nu^2}{2}(\phi(s,y))^2 + 1 & ; \quad \phi(0,y) = y \\ \dot\xi(s,y) = \eta\phi(s,y) & ; \quad \xi(0,y) = 0 . \end{cases}$$

Explicitly,

$$\begin{aligned} \phi(s,y) &= \frac{1 + D(y)e^{-A(y)s}}{B + C(y)e^{-A(y)s}} , \\ \xi(s,y) &= \frac{\eta}{B}\left\{\frac{C(y) - BD(y)}{A(y)C(y)} \log \frac{B + C(y)e^{-A(y)s}}{B + C(y)} + s\right\} . \end{aligned}$$

where A, B, C and D are given by

$$B = \frac{1}{2}\left(\eta + \sqrt{\eta^2 + 2\nu^2}\right) , \quad C(y) = (1 - By)\frac{\eta + \nu^2 y - \sqrt{\eta^2 + 2\nu^2}}{2\eta y + \nu^2 y - 2} ,$$

$$D(y) = (B + C(y))y - 1 , \quad A(y) = \frac{-C(y)(2B - \eta) + D(y)(\nu^2 + \eta B)}{BD(y) - C(y)} .$$

In the following proposition (see also Shirakawa [28]), we generalize Lemma A.1 to the case of a piecewise constant function $\mu(\cdot)$. We denote $T_0 = 0$ and $\Delta_j = T_j - T_{j-1}$. The functions ϕ and ξ are those of Lemma A.1.

Proposition A.2 *Assume that $\mu(\cdot)$ is a piecewise constant function: $\mu(t) = \mu_k$ on $t \in [T_{k-1}, T_k]$ for $k = 1, ..., m$. For $t < s$, let $i \leq j$ such that $t \in [T_{i-1}, T_i)$ and $s \in (T_j, T_{j+1}]$. Then*
(i) *For any $x \geq 0$ and $y \geq 0$,*

$$\begin{aligned}\widetilde{\mathcal{E}}(s,t,x,y) &:= \mathbb{E}\Big(\exp(-\int_t^s X_u du - yX_s)|X_t = x\Big) = \\ &\exp\Big\{-\mu_i\xi(T_i - t, y_i) - x\phi(T_i - t, y_i) - \sum_{k=i+1}^{j}\mu_k\xi(\Delta_k, y_k) - \mu_{j+1}\xi(s - T_j, y)\Big\}\end{aligned} \tag{36}$$

with

$$\begin{aligned}y_j &= y_j(s) := \phi(s - T_j, y)\ , \\ y_k &= y_k(s) := \phi(\Delta_{k+1}, y_{k+1}(s))\ , \quad k < j.\end{aligned} \tag{37}$$

(ii) *One has,*

$$\mathbb{E}\Big(X_s \exp\big(-\int_t^s X_u du\big)|X_t\Big) = \widetilde{\mathcal{D}}(s,t,X_t)\mathbb{E}\Big(\exp\big(-\int_t^s X_u du\big)|X_t\Big)$$

where

$$\begin{aligned}\widetilde{\mathcal{D}}(s,t,x) &= \mu_i\frac{\partial\xi}{\partial y}(T_i - t, y_i)\frac{dy_i}{ds} + x\frac{\partial\phi}{\partial y}(T_i - t, y_i)\frac{dy_i}{ds} \\ &+ \sum_{k=i+1}^{j}\mu_k\frac{\partial\xi}{\partial y}(\Delta_k, y_k)\frac{dy_k}{ds} + \mu_{j+1}\frac{\partial\xi}{\partial s}(s - T_j, 0),\end{aligned}$$

and the $y_k s$ are as in (37) with $y = 0$.

Setting $t = 0$ and $s = T_j$ in the first part of the above proposition, one obtains:

Corollary A.3 *One has*

$$\mathbb{E}\Big(\exp\big(-\int_0^{T_j} X_u du\big)\Big) = \exp\left(-a_{j,0}X_0 - \sum_{k=1}^{j}\mu_k\xi(\Delta_k, a_{j,k})\right)$$

with

$$a_{j,j} = 0 \quad , \quad a_{j,k} = \phi(\Delta_{k+1}, a_{j,k+1}) \quad \textit{for} \;\; k < j\ . \tag{38}$$

References

[1] Assefa, S., Bielecki, T.R., Crépey, S. and Jeanblanc, M.: CVA computation for counterparty risk assessment in credit portfolios, Forthcoming in *Credit Risk Frontiers: Subprime crisis, Pricing and Hedging, CVA, MBS, Ratings and Liquidity*, Bielecki, T.R., Brigo, D. and Patras, F. eds, Wiley.

[2] Basel Committee on Banking Supervision: The Application of Basel II to Trading Activities and the Treatment of Double Default Effects, *Bank for International Settlements*, July 2005.

[3] Basel Committee on Banking Supervision: International Convergence of Capital Measurement and Capital Standards, *Bank for International Settlements*, June 2006.

[4] Bielecki, T.R. and Rutkowski, M.: *Credit Risk: Modeling, Valuation and Hedging*, Springer-Verlag, Berlin, 2002.

[5] Bielecki, T.R., Vidozzi, A. and Vidozzi, L.: A Markov Copulae Approach to Pricing and Hedging of Credit Index Derivatives and Ratings Triggered Step–Up Bonds, *Journal of Credit Risk*, Vol. 4, No. 1, 2008.

[6] Bielecki, T.R., Jeanblanc, M., and Rutkowski, M.: Pricing and Trading Credit Default Swaps in a hazard process model, *Annals of Applied Probability*, Vol. 18, No. 6, pp. 2495–2529, 2008.

[7] Bielecki, T.R., Cialenco, I., and Iyigunler, I.: Counterparty Risk and the Impact of Collateralization in CDS Contracts, *Working Paper* , 2010.

[8] Blanchet-Scalliet, Ch., Patras, F.: Structural Counterparty Risk Valuation for CDS, Forthcoming in *Credit Risk Frontiers: Subprime crisis, Pricing and Hedging, CVA, MBS, Ratings and Liquidity*, Bielecki, T.R., Brigo, D. and Patras, F. eds, Wiley.

[9] D. Brigo: Candidate Market Models and the Calibrated CIR++ Stochastic Intensity Model for Credit Default Swap Options and Callable Floaters, *Risk Magazine*, 2005.

[10] Brigo, D., Capponi, A.: Bilateral counterparty risk valuation with stochastic dynamical models and application to Credit Default Swaps, *Working Paper*, 2008.

[11] Brigo, D., Chourdakis, K.: Counterparty Risk for Credit Default Swaps: Impact of spread volatility and default correlation, *International Journal of Theoretical and Applied Finance*, Vol. 12, No. 7, pp. 1007–1026, 2009.

[12] Brigo, D., Masetti, M.: Risk Neutral Pricing of Counterparty Risk, *Counterparty Credit Risk Modeling: Risk Management, Pricing and Regulation*, ed. Pykhtin, M., Risk Books, London, 2006.

[13] Cesari G., Aquilina J. and Charpillon N.: *Modelling, Pricing, and Hedging Counterparty Credit Exposure*, Springer Finance, 2010.

[14] Cesari G.: *Personal communication.*

[15] Crépey, S., Jeanblanc, M. and Zargari, B.: Counterparty Risk on a CDS in a Markov Chain Copula Model with Joint Defaults, *Recent Advances in Financial Engineering 2009*, M. Kijima, C. Hara, Y. Muromachi and K. Tanaka, eds., World Scientific Publishing Co. Pte., pp. 91–126, 2010. available on `http://grozny.maths.univ-evry.fr/pages_perso/crepey/`.

[16] Dellacherie, C., Meyer, P.-A.: Probabilité et Potentiel. Vol. I, Hermann, 1975.

[17] Duffie, D. and Gârleanu, N.: Risk and the valuation of collateralized debt obligations, *Financial Analysts Journal*, Vol. 57, No. 1, pp. 41–62, 2001.

[18] Ethier, H.J. and Kurtz, T.G.: *Markov Processes, Characterization and Convergence*. Wiley, 1986.

[19] Gregory J.: *Counterparty Credit Risk: The New Challenge for Global Financial Markets*, Wiley, 2009.

[20] Hull, J., White, A.: Valuing Credit Default Swaps II: Modeling Default Correlation, *The Journal of Derivatives*, Vol. 8, No. 3, pp. 12–22, 2001.

[21] Huge, B. and Lando, D.: Swap Pricing with Two-Sided Default Risk in a Rating-Based Model, *European Finance Review*, Vol. 3, pp. 239–268, 1999.

[22] Jarrow, R. and Yu, F.: Counterparty risk and the pricing of defaultable securities, *Journal of Finance*, Vol. 56, pp. 1765–1799, 2001.

[23] Karatzas, I. and Shreve, S.. *Brownian Motion and Stochastic Calculus*, Springer-Verlag, 1988.

[24] Lardy, J.P.: Counterparty Risk on CDS: A composite Spread Method example, *CRIS Seminar*, 2008.

[25] Lipton, A. and Sepp, A.: Credit value adjustment for credit default swaps via the structural default model, *The Journal of Credit Risk*, Vol. 5, No. 2, pp. 123-146, 2009.

[26] Leung, S.Y., Kwok, Y.K.: Credit Default Swap Valuation with Counterparty Risk, *Kyoto Economic Review*, Vol. 74, No. 1, pp. 25–45, 2005.

[27] Redon, C.: Wrong-way risk modelling, *Risk magazine*, April 2006.

[28] Shirakawa, H.: A Squared Bessel Processes and Their Applications to the Square Root Interest Rate Model, *Financial Engineering and the Japanese Markets*, Vol. 9, No. 3-4, pp. 169–190, 2002.

[29] Stein, H. and Lee, K.P.: Counterparty Valuation Adjustments, Forthcoming in *Credit Risk Frontiers: Subprime crisis, Pricing and Hedging, CVA, MBS, Ratings and Liquidity*, Bielecki, T.R., Brigo, D. and Patras, F., eds, Wiley.