On \( \mathbb{R}^d \)-valued peacocks

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Abstract: In this paper, we consider \( \mathbb{R}^d \)-valued integrable processes which are increasing in the convex order, i.e. \( \mathbb{R}^d \)-valued peacocks in our terminology. After the presentation of some examples, we show that an \( \mathbb{R}^d \)-valued process is a peacock if and only if it has the same one-dimensional marginals as an \( \mathbb{R}^d \)-valued martingale. This extends former results, obtained notably by V. Strassen (1965), J.L. Doob (1968) and H. Kellerer (1972).

Key words: convex order; martingale; 1-martingale; peacock.


1 Introduction

1.1 Terminology

First we fix the terminology. In the sequel, \( d \) denotes a fixed integer and \( \mathbb{R}^d \) is equipped with a norm which is denoted by \( | \cdot | \).

We say that two \( \mathbb{R}^d \)-valued processes: \( (X_t, t \geq 0) \) and \( (Y_t, t \geq 0) \) are associated, if they have the same one-dimensional marginals, i.e. if:

\[ \forall t \geq 0, \quad X_t \overset{\text{law}}{=} Y_t. \]

A process which is associated with a martingale is called a \( 1 \)-martingale.

An \( \mathbb{R}^d \)-valued process \( (X_t, t \geq 0) \) will be called a peacock if:
i) it is integrable, that is:
\[ \forall t \geq 0, \quad \mathbb{E}[|X_t|] < \infty \]

ii) it increases in the convex order, meaning that, for every convex function
\( \psi : \mathbb{R}^d \rightarrow \mathbb{R} \), the map:
\[ t \geq 0 \rightarrow \mathbb{E}[\psi(X_t)] \in (-\infty, +\infty] \]
is increasing.

This terminology was introduced in [HPRY]. We refer the reader to this monograph for an explanation of the origin of the term: "peacock", as well as for a comprehensive study of this notion in the case \( d = 1 \).

Actually, it may be noted that, in the definition of a peacock, only the family \((\mu_t, t \geq 0)\) of its one-dimensional marginals is involved. This makes it natural, in the following, to also call a peacock, a family \((\mu_t, t \geq 0)\) of probability measures on \( \mathbb{R}^d \) such that:

i) \( \forall t \geq 0, \quad \int |x| \mu_t(dx) < \infty \),

ii) for every convex function \( \psi : \mathbb{R}^d \rightarrow \mathbb{R} \), the map:
\[ t \geq 0 \rightarrow \int \psi(x) \mu_t(dx) \in (-\infty, +\infty] \]
is increasing.

Likewise, a family \((\mu_t, t \geq 0)\) of probability measures on \( \mathbb{R}^d \) and an \( \mathbb{R}^d\)-valued process \((Y_t, t \geq 0)\) will be said to be associated if, for every \( t \geq 0 \), the law of \( Y_t \) is \( \mu_t \), i.e. if \((\mu_t, t \geq 0)\) is the family of the one-dimensional marginals of \((Y_t, t \geq 0)\).

Obviously, the above notions also are meaningful if one considers processes and families of measures indexed by a subset of \( \mathbb{R}_+ \) (for example \( \mathbb{N} \)) instead of \( \mathbb{R}_+ \).

It is an easy consequence of Jensen’s inequality that an \( \mathbb{R}^d\)-valued process which is a 1-martingale, is a peacock. So, a natural question is whether the converse holds.
1.2 Case $d = 1$

A remarkable result due to H. Kellerer ([K], 1972) states that, actually, any $\mathbb{R}$-valued process which is a peacock, is a 1-martingale. More precisely, Kellerer’s result states that any $\mathbb{R}$-valued peacock admits an associated martingale which is Markovian.

Two more recent results now complete Kellerer’s theorem.

i) G. Lowther ([L], 2008) states that if $(\mu_t, t \geq 0)$ is an $\mathbb{R}$-valued peacock such that the map: $t \mapsto \mu_t$ is weakly continuous (i.e. for any $\mathbb{R}$-valued, bounded and continuous function $f$ on $\mathbb{R}$, the map: $t \mapsto \int f(x) \mu_t(dx)$ is continuous), then $(\mu_t, t \geq 0)$ is associated with a strongly Markovian martingale which moreover is “almost-continuous” (see [L] for the definition).

ii) In a previous paper ([HR], 2011), we presented a new proof of the above mentioned theorem of H. Kellerer. Our method, which is inspired from the “Fokker-Planck Equation Method” ([HPRY, Section 6.2, p.229]), then appears as a new application of M. Pierre’s uniqueness theorem for a Fokker-Planck equation ([HPRY, Theorem 6.1, p.223]). Thus, we show that a martingale which is associated to an $\mathbb{R}$-valued peacock, may be obtained as a limit of solutions of stochastic differential equations. However, we do not obtain that such a martingale is Markovian.

1.3 Case $d \geq 1$

Concerning the case $\mathbb{R}^d$ with $d \geq 1$, and even much more general spaces, we would like to mention the following three important papers.

i) In [CFM] (1964), P. Cartier, J.M.G. Fell and P.-A. Meyer study the case of two probability measures $(\mu_1, \mu_2)$ on a metrizable convex compact $K$ of a locally convex space. They prove, using the Hahn-Banach theorem, that, if $(\mu_1, \mu_2)$ is a $K$-valued peacock (indexed by $\{1, 2\}$), then there exists a Markovian kernel $P$ on $K$ such that: $\theta(dx_1, dx_2) := \mu_1(dx_1) P(x_1, dx_2)$ is the law of a $K$-valued martingale $(Y_1, Y_2)$ associated to $(\mu_1, \mu_2)$.

ii) In [S] (1965), V. Strassen extends the Cartier-Fell-Meyer result to $\mathbb{R}^d$-valued peacocks without making the assumption of compact support. Then he proves that, if $(\mu_n, n \geq 0)$ is an $\mathbb{R}^d$-valued peacock (indexed by $\mathbb{N}$), there exists an associated martingale which is obtained as a Markov chain.
iii) In [D] (1968), J.L. Doob studies, in a very general extended framework, peacocks indexed by \( \mathbb{R}_+ \) and taking their values in a fixed compact set. In particular, he proves that they admit associated martingales. Note that in [D], the Markovian character of the associated martingales is not considered.

### 1.4 Organization

The remainder of this paper is organised as follows:

- In Section 2, we present some basic facts concerning the \( \mathbb{R}^d \)-valued peacocks and we describe some examples, thus extending results of [HPRY].

- In Section 3, starting from Strassen’s theorem, we prove that a family \((\mu_t, t \geq 0)\) of probability measures on \( \mathbb{R}^d \), is associated to a right-continuous martingale, if and only if, \((\mu_t, t \geq 0)\) is a peacock such that the map: \( t \rightarrow \mu_t \) is weakly right-continuous on \( \mathbb{R}_+ \).

- In Section 4, by approximation from the previous result, we extend this result to the case of general \( \mathbb{R}^d \)-valued peacocks.

### 2 Generalities, Examples

#### 2.1 Notation

In the sequel, \( d \) denotes a fixed integer, \( \mathbb{R}^d \) is equipped with a norm which is denoted by \( \cdot \), and we adopt the terminology of Subsection 1.1.

We also denote by \( \mathcal{M} \) the set of probability measures on \( \mathbb{R}^d \), equipped with the topology of weak convergence (with respect to the space \( C_b(\mathbb{R}^d) \) of \( \mathbb{R} \)-valued, bounded, continuous functions on \( \mathbb{R}^d \)). We denote by \( \mathcal{M}_f \) the subset of \( \mathcal{M} \) consisting of measures \( \mu \in \mathcal{M} \) such that \( \int |x| \mu(dx) < \infty \). \( \mathcal{M}_f \) is also equipped with the topology of weak convergence.

\( C_c(\mathbb{R}^d) \) denotes the space of \( \mathbb{R} \)-valued continuous functions on \( \mathbb{R}^d \) with compact support, and \( C_c^+(\mathbb{R}^d) \) is the subspace consisting of all the nonnegative functions in \( C_c(\mathbb{R}^d) \).
2.2 Basic facts

**Proposition 2.1** Let \((X_t, t \geq 0)\) be an \(\mathbb{R}^d\)-valued integrable process. Then \((X_t, t \geq 0)\) is a peacock if (and only if) the map: \(t \mapsto \mathbb{E}[\psi(X_t)]\) is increasing, for every function \(\psi : \mathbb{R}^d \to \mathbb{R}\) which is convex, of \(C^\infty\) class and such that the derivative \(\psi'\) is bounded on \(\mathbb{R}^d\).

**Proof** Let \(\psi : \mathbb{R}^d \to \mathbb{R}\) be a convex function. For every \(a \in \mathbb{R}^d\), there exists an affine function \(h_a\) such that:

\[\forall x \in \mathbb{R}^d, \quad \psi(x) \geq h_a(x)\quad \text{and} \quad \psi(a) = h_a(a).\]

Let \(\{a_n; n \geq 1\}\) be a countable dense subset of \(\mathbb{R}^d\). We set:

\[\forall n \geq 1, \quad \psi_n(x) = \sup_{1 \leq j \leq n} h_{a_j}(x).\]

Then:

\[\forall x \in \mathbb{R}^d, \quad \lim_{n \uparrow \infty} \psi_n(x) = \psi(x).\]

The functions \(\psi_n\) are convex and Lipschitz continuous.

Let \(\phi\) be a nonnegative function, of \(C^\infty\) class, with compact support and such that \(\int \phi(x) \, dx = 1\). We set, for \(n, p \geq 1\),

\[\forall x \in \mathbb{R}^d, \quad \psi_{n,p}(x) = \int \psi_n \left( x - \frac{1}{p} y \right) \phi(y) \, dy.\]

Clearly, \(\psi_{n,p}\) is convex, of \(C^\infty\) class and Lipschitz continuous. Consequently, its derivative is bounded on \(\mathbb{R}^d\). Moreover, \(\lim_{p \to \infty} \psi_{n,p} = \psi_n\) uniformly on \(\mathbb{R}^d\).

The desired result now follows directly.

\(\square\)

The next result will be useful in the sequel.

**Proposition 2.2** Let \((X_t, t \geq 0)\) be an \(\mathbb{R}^d\)-valued peacock. Then:

1. the map: \(t \mapsto \mathbb{E}[X_t]\) is constant;
2. the map: \(t \mapsto \mathbb{E}[[X_t]]\) is increasing, and therefore, for every \(T \geq 0\),

\[\sup_{0 \leq t \leq T} \mathbb{E}[[X_t]] = \mathbb{E}[[X_T]] < \infty;\]
3. for every \( T \geq 0 \), the random variables \( (X_t ; 0 \leq t \leq T) \) are uniformly integrable.

**Proof** Properties 1 and 2 are obvious.

If \( c \geq 0 \),

\[
|x| 1_{|x| \geq c} \leq (2|x| - c)^+ .
\]

As the function \( x \rightarrow (2|x| - c)^+ \) is convex,

\[
\sup_{t \in [0,T]} \mathbb{E} [ |X_t| 1_{|X_t| \geq c} ] \leq \mathbb{E}[(2|X_T| - c)^+] .
\]

Now, by dominated convergence,

\[
\lim_{c \to +\infty} \mathbb{E}[(2|X_T| - c)^+] = 0 .
\]

Hence, property 3 holds.

\qed

### 2.3 Examples

The following examples are given in [HPRY] for \( d = 1 \). The proofs given below are essentially the same as in [HPRY].

**Proposition 2.3** Let \( X \) be a centered \( \mathbb{R}^d \)-valued random variable. Then \( (tX , t \geq 0) \) is a peacock.

**Proof** Let \( \psi : \mathbb{R}^d \rightarrow \mathbb{R} \) be a convex function, and \( 0 \leq s < t \). Then,

\[
\psi(sX) \leq \left(1 - \frac{s}{t}\right) \psi(0) + \frac{s}{t} \psi(tX) .
\]

Since \( X \) is centered, by Jensen’s inequality:

\[
\psi(0) = \psi(\mathbb{E}[tX]) \leq \mathbb{E}[\psi(tX)] .
\]

Hence,

\[
\mathbb{E}[\psi(sX)] \leq \left(1 - \frac{s}{t}\right) \mathbb{E}[\psi(tX)] + \frac{s}{t} \mathbb{E}[\psi(tX)] = \mathbb{E}[\psi(tX)] .
\]

\qed
**Proposition 2.4** Let \((X_t, t \geq 0)\) be a family of centered, \(\mathbb{R}^d\)-valued, Gaussian variables. We denote by \(C(t) = (c_{i,j}(t))_{1 \leq i,j \leq d}\) the covariance matrix of \(X_t\). Then, \((X_t, t \geq 0)\) is a peacock if and only if the map: 
\(t \mapsto C(t)\) is increasing in the sense of quadratic forms, i.e:
\[
\forall a = (a_1, \ldots, a_d) \in \mathbb{R}^d, \quad t \mapsto \sum_{1 \leq i,j \leq d} c_{i,j}(t) a_i a_j \quad \text{is increasing.}
\]

**Proof**

1) For every \(a \in \mathbb{R}^d\), the function:
\[
x \in \mathbb{R}^d \mapsto \sum_{1 \leq i,j \leq d} a_i a_j x_i x_j = \left( \sum_{i=1}^{d} a_i x_i \right)^2
\]
is convex. This entails that, if \((X_t, t \geq 0)\) is a peacock, then the map: 
\(t \mapsto C(t)\) is increasing in the sense of quadratic forms.

2) Conversely, suppose that the map: \(t \mapsto C(t)\) is increasing in the sense of quadratic forms. By the proof of [HPRY, Theorem 2.16, p.132], there exists a centered \(\mathbb{R}^d\)-valued Gaussian process: \((\Gamma_t = (\Gamma_{1,t}, \ldots, \Gamma_{d,t}), t \geq 0)\), such that:
\[
\forall s,t \geq 0, \; \forall 1 \leq i,j \leq d, \quad E[\Gamma_{i,s} \Gamma_{j,t}] = c_{i,j}(s \wedge t).
\]

Therefrom we deduce that \((\Gamma_t, t \geq 0)\) is a martingale which is associated to \((X_t, t \geq 0)\), and consequently, \((X_t, t \geq 0)\) is a peacock.

\(\Box\)

**Corollary 2.1** Let \(A\) be a \(d \times d\) matrix. We consider the \(\mathbb{R}^d\)-valued Ornstein-Uhlenbeck process \((U_t, t \geq 0)\), defined as (the unique) solution, started from 0, of the SDE:
\[
dU_t = dB_t + AU_t \, dt
\]
where \((B_t, t \geq 0)\) denotes a \(d\)-dimensional Brownian motion. Then, \((U_t, t \geq 0)\) is a peacock.
Proof. One has:
\[ U_t = \int_0^t \exp((t - s)A) \, dB_s. \]
Hence, for every \( t \geq 0 \), \( U_t \) is a centered, \( \mathbb{R}^d \)-valued Gaussian variable whose covariance matrix is:
\[ C(t) = \int_0^t \exp(sA) \exp(sA^*) \, ds \]
where \( A^* \) denotes the adjoint matrix of \( A \). Therefrom it is clear that the map: \( t \rightarrow C(t) \) is increasing in the sense of quadratic forms, and Proposition 2.4 applies.

\[ \square \]

Proposition 2.5 Let \((M_t, \ t \geq 0)\) be an \( \mathbb{R}^d \)-valued, right-continuous martingale such that:
\[ \forall T > 0, \quad \mathbb{E} \left[ \sup_{0 \leq t \leq T} |M_t| \right] < \infty. \]
Then,
1. \((X_t := \frac{1}{t} \int_0^t M_s \, ds ; \ t \geq 0)\) is a peacock,
2. \((\tilde{X}_t := \int_0^t (M_s - M_0) \, ds ; \ t \geq 0)\) is a peacock.

Proof. Using Proposition 2.1, we may use the proof of [HPRY, Theorem 1.4, p.26]. For the convenience of the reader, we reproduce this proof below.

1) Let \( \psi : \mathbb{R}^d \rightarrow \mathbb{R} \) be a convex function, of \( C^\infty \) class and such that the derivative \( \psi' \) is bounded on \( \mathbb{R}^d \). Setting:
\[ \tilde{M}_t = \int_0^t s \, dM_s, \]
one has, by integration by parts:
\[ X_t = M_t - t^{-1} \tilde{M}_t \quad \text{and} \quad dX_t = t^{-2} \tilde{M}_t \, dt. \]
Denoting by \( \mathcal{F}_s \) the \( \sigma \)-algebra generated by \( \{M_u ; 0 \leq u \leq s\} \), one gets, for \( 0 \leq s \leq t \),
\[ \mathbb{E}[X_t | \mathcal{F}_s] = X_s + (s^{-1} - t^{-1}) \tilde{M}_s. \]
Consequently, by Jensen’s inequality,
\[
\mathbb{E}[\psi(X_t)] \geq \mathbb{E}[\psi(X_s + (s^{-1} - t^{-1}) \hat{M}_s)] .
\]
Using again the fact that \( \psi \) is convex, one obtains:
\[
\mathbb{E}[\psi(X_t)] \geq \mathbb{E}[\psi(X_s)] + (s^{-1} - t^{-1}) \mathbb{E}[\psi'(X_s) \cdot \hat{M}_s] .
\]
Now,
\[
\psi'(X_s) \cdot \hat{M}_s = \int_0^s u^{-2} \psi''(X_u) (\hat{M}_u, \hat{M}_u) \, du + \int_0^s u \psi'(X_u) \cdot dM_u
\]
and therefore
\[
\mathbb{E}[\psi(X_t)] - \mathbb{E}[\psi(X_s)] \geq (s^{-1} - t^{-1}) \mathbb{E}[\psi'(X_s) \cdot \hat{M}_s] \geq 0 ,
\]
which, by Proposition 2.1, yields the desired result.

2) Let \( \psi \) be as above. One may suppose that \( M_0 = 0 \). One has, for \( 0 \leq s \leq t \),
\[
\mathbb{E}[\tilde{X}_t \mid \mathcal{F}_s] = \tilde{X}_s + (t - s) M_s .
\]
Consequently, by Jensen’s inequality,
\[
\mathbb{E}[\psi(\tilde{X}_t)] \geq \mathbb{E}[\psi(\tilde{X}_s + (t - s) M_s)] .
\]
Using again the fact that \( \psi \) is convex, one obtains:
\[
\mathbb{E}[\psi(\tilde{X}_t)] \geq \mathbb{E}[\psi(\tilde{X}_s)] + (t - s) \mathbb{E}[\psi'(\tilde{X}_s) \cdot M_s] .
\]
Now,
\[
\psi'(\tilde{X}_s) \cdot M_s = \int_0^s \psi''(\tilde{X}_u) (M_u, M_u) \, du + \int_0^s \psi'(\tilde{X}_u) \cdot dM_u
\]
and therefore
\[
\mathbb{E}[\psi(\tilde{X}_t)] - \mathbb{E}[\psi(\tilde{X}_s)] \geq (t - s) \mathbb{E}[\psi'(\tilde{X}_s) \cdot M_s] \geq 0 ,
\]
which, by Proposition 2.1, yields the desired result.

\( \square \)
3 Right-continuous peacocks

In this section, we shall show that any right continuous peacock admits an associated right-continuous martingale. For this, we start from Strassen’s theorem, which we now recall.

**Theorem 3.1 (Strassen [S], Theorem 8)** Let \((\mu_n, n \in \mathbb{N})\) be a sequence in \(\mathcal{M}\). Then \((\mu_n, n \in \mathbb{N})\) is a peacock if and only if there exists a martingale \((M_n, n \in \mathbb{N})\) which is associated to \((\mu_n, n \in \mathbb{N})\).

We shall extend this theorem to right-continuous peacocks indexed by \(\mathbb{R}_+\). In the case \(d = 1\), the following theorem is proven in [HR], by a quite different method. In particular, in [HR], we do not use Strassen’s theorem, nor the Hahn-Banach theorem, but an explicit approximation by solutions of SDE’s.

**Theorem 3.2** Let \((\mu_t, t \geq 0)\) be a family in \(\mathcal{M}\). Then the following properties are equivalent:

i) There exists a right-continuous martingale associated to \((\mu_t, t \geq 0)\).

ii) \((\mu_t, t \geq 0)\) is a peacock and the map:

\[
 t \geq 0 \longrightarrow \mu_t \in \mathcal{M}
\]

is right-continuous.

**Proof**

1) We first assume that property i) is satisfied. Then, the fact that \((\mu_t, t \geq 0)\) is a peacock follows classically from Jensen’s inequality. Let \((M_t, t \geq 0)\) be a right-continuous martingale associated to \((\mu_t, t \geq 0)\). Then, if \(f \in C_b(\mathbb{R}^d)\), dominated convergence yields that, for any \(t \geq 0\),

\[
 \lim_{s \to t, s > t} \int f(x) \mu_s(dx) = \lim_{s \to t, s > t} \mathbb{E}[f(M_s)] = \mathbb{E}[f(M_t)] = \int f(x) \mu_t(dx).
\]

Therefore, the map:

\[
 t \geq 0 \longrightarrow \mu_t \in \mathcal{M}
\]

is right-continuous, and property ii) is satisfied.

2) Conversely, we now assume that property ii) is satisfied. For every \(n \in \mathbb{N}\), we set:

\[
 \mu_k^{(n)} = \mu_{k2^{-n}}, \quad k \in \mathbb{N}.
\]
By Strassen’s theorem (Theorem 3.1), there exists a martingale \((M_k^{(n)}, k \in \mathbb{N})\) which is associated to \((\mu_k^{(n)}, k \in \mathbb{N})\). We set:

\[X_t^{(n)} = M_k^{(n)} \text{ if } t = k2^{-n} \quad \text{and} \quad X_t^{(n)} = 0 \text{ otherwise.}\]

Consequently, the law of \(X_t^{(n)}\) is \(\mu_t\) if \(t \in \{k2^{-n}; k \in \mathbb{N}\}\), and is \(\delta\) (the Dirac measure at 0) if \(t \not\in \{k2^{-n}; k \in \mathbb{N}\}\).

Note that, due to the lack of uniqueness in Strassen’s theorem, the law of \((X_{k2^{-n}}^{(n)}, k \in \mathbb{N})\) may be not the same as the law of \((X_{k2^{-n}}^{(n+1)}, k \in \mathbb{N})\). Only the one-dimensional marginals are identical.

3) Let \(D = \{k2^{-n}; k, n \in \mathbb{N}\}\) the set of dyadic numbers. For every \(n \in \mathbb{N}\), for every \(r \geq 1\) and \(\tau_r = (t_1, t_2, \cdots, t_r) \in D^r\), we denote by \(\Pi_{\tau_r}^{(r,n)}\) the law of \((X_{t_1}^{(n)}, \cdots, X_{t_r}^{(n)})\), a probability on \((\mathbb{R}^d)^r\).

**Lemma 3.1** For every \(\tau_r \in D^r\), the set of probability measures: \(\{\Pi_{\tau_r}^{(r,n)}; n \in \mathbb{N}\}\) is tight.

**Proof** We set, for \(x = (x^1, \cdots, x^r) \in (\mathbb{R}^d)^r\), \(|x|_r = \sum_{j=1}^{r} |x^j|\). Then, for \(p > 0\),

\[\Pi_{\tau_r}^{(r,n)}(|x|_r \geq p) \leq \frac{1}{p} \Pi_{\tau_r}^{(r,n)}(|x|_r) = \frac{1}{p} \sum_{j=1}^{r} \mathbb{E}[|X_{t_j}^{(n)}|] \leq \frac{1}{p} \sum_{j=1}^{r} \mu_{t_j}(|x|)\]

since, by point 2), the law of \(X_{t_j}^{(n)}\) is either \(\mu_{t_j}\) or \(\delta\). Hence,

\[\lim_{p \to \infty} \sup_{n \geq 0} \Pi_{\tau_r}^{(r,n)}(|x|_r \geq p) = 0.\]

4) As a consequence of the previous lemma, and with the help of the diagonal procedure, there exists a subsequence \((n_l)_{l \geq 0}\) such that, for every \(\tau_r \in D^r\), the sequence of probabilities on \((\mathbb{R}^d)^r\): \((\Pi_{\tau_r}^{(r,n_l)}), l \geq 0\), weakly converges to a probability which we denote by \(\Pi_{\tau_r}^{(r)}\). We remark that, for \(l\) large enough, the law of \(X_{t_j}^{(n_l)}\) is \(\mu_{t_j}\). Then, there exists an \(\mathbb{R}^d\)-valued process \((X_t, t \in D)\) such that, for every \(r \in \mathbb{N}\) and every \(\tau_r = (t_1, \cdots, t_r) \in D^r\), the law of \((X_{t_1}, \cdots, X_{t_r})\) is \(\Pi_{\tau_r}^{(r)}\), and \(\Pi_{t}^{(1)} = \mu_t\) for every \(t \in D\).
Lemma 3.2 The process \((X_t, t \in D)\) is a martingale associated to \((\mu_t, t \in D)\).

Proof As we have already seen, the process \((X_t, t \in D)\) is associated to \((\mu_t, t \in D)\). We now prove that it is a martingale. We set:

\[ \forall p > 0, \forall x \in \mathbb{R}^d, \quad \varphi_p(x) = \left(1 \lor \frac{|x|}{p}\right)^{-1} x. \]

Then,

\[ \varphi_p \in C_b(\mathbb{R}^d; \mathbb{R}^d) \quad \text{and} \quad \varphi_p(x) = x \quad \text{for} \quad |x| \leq p. \]

Let \(0 \leq s_1 < s_2 < \cdots < s_r \leq s \leq t\) be elements of \(D\), and let \(f \in C_b((\mathbb{R}^d)^r)\). We set: \(\|f\|_\infty = \sup\{|f(x)| ; x \in (\mathbb{R}^d)^r\}\). Then, for \(l\) large enough,

\[ \mathbb{E}[f(X_s^{(n_1)}, \ldots, X_s^{(n_i)}) X_t^{(n_i)}] = \mathbb{E}[f(X_{s_1}^{(n_1)}, \ldots, X_{s_r}^{(n_i)}) X_s^{(n_i)}]. \]

On the other hand,

\[ |\mathbb{E}[f(X_{s_1}, \ldots, X_{s_r}) \varphi_p(X_t)] - \mathbb{E}[f(X_{s_1}, \ldots, X_{s_r}) X_t]| \leq \|f\|_\infty \mu_t \left(|x| 1_{\{|x| \geq p\}}\right), \quad \text{for every} \quad p > 0, \]

and likewise, replacing \(t\) by \(s\). Moreover,

\[ \lim_{l \to \infty} \mathbb{E}[f(X_{s_1}^{(n_1)}, \ldots, X_{s_r}^{(n_i)}) \varphi_p(X_t^{(n_i)})] = \mathbb{E}[f(X_{s_1}, \ldots, X_{s_r}) \varphi_p(X_t)], \]

and likewise, replacing \(t\) by \(s\). Finally, we obtain, for \(p > 0,\)

\[ |\mathbb{E}[f(X_{s_1}, \ldots, X_{s_r}) X_t] - \mathbb{E}[f(X_{s_1}, \ldots, X_{s_r}) X_s]| \leq 2 \|f\|_\infty \left[\mu_t \left(|x| 1_{\{|x| \geq p\}}\right) + \mu_s \left(|x| 1_{\{|x| \geq p\}}\right)\right], \]

and the desired result follows, letting \(p\) go to \(\infty\). 

\[ \square \]
5) By the classical theory of martingales (see, for example, [DM]), almost surely, for every \( t \geq 0 \),
\[
M_t = \lim_{s \to t, s \in D, s > t} X_s
\]
is well defined, and \((M_t, t \geq 0)\) is a right-continuous martingale. Besides, since, by hypothesis, the map: \( t \geq 0 \mapsto \mu_t \in \mathcal{M} \) is right-continuous, we deduce from Lemma 3.2 that this martingale \((M_t, t \geq 0)\) is associated to \((\mu_t, t \geq 0)\).

\[ \square \]

4 The general case

Theorem 3.2 shall now be extended, by approximation, to the general case.

**Theorem 4.1** Let \((\mu_t, t \geq 0)\) be a family in \( \mathcal{M} \). Then the following properties are equivalent:

i) There exists a martingale associated to \((\mu_t, t \geq 0)\).

ii) \((\mu_t, t \geq 0)\) is a peacock.

**Proof** Let \((\mu_t, t \geq 0)\) be a peacock.

**Lemma 4.1** There exists a countable set \( \Delta \subset \mathbb{R}_+ \) such that the map:
\[
t \mapsto \mu_t \in \mathcal{M}
\]
is continuous at any \( s \not\in \Delta \).

**Proof** Let \( \chi : \mathbb{R}^d \to \mathbb{R}_+ \) be defined by:
\[
\chi(x) = (1 - |x|)^+ = (1 \lor |x|) - |x|.
\]
Then \( \chi \in C^+_c(\mathbb{R}^d) \) and \( \chi \) is the difference of two convex functions. We set:
\[
\chi_m(x) = m^d \chi(mx),
\]
and we define the countable set \( \mathcal{H} \) by:
\[
\mathcal{H} = \left\{ \sum_{j=0}^{r} a_j \chi_m(x - q_j) ; r \in \mathbb{N}, m \in \mathbb{N}, a_j \in \mathbb{Q}_+, q_j \in \mathbb{Q}^d \right\}.
\]
For $h \in H$, the function: $t \mapsto \mu_t(h)$ is the difference of two increasing functions, and hence admits a countable set $\Delta_h$ of discontinuities. We set $\Delta = \bigcup_{h \in H} \Delta_h$. Then $\Delta$ is a countable subset of $\mathbb{R}_+$, and $t \mapsto \mu_t(h)$ is continuous at any $s \not\in \Delta$, for every $h \in H$. Now, it is easy to see that $H$ is dense in $C_c^+(\mathbb{R}^d)$ in the following sense: for every $\varphi \in C_c^+(\mathbb{R}^d)$, there exist a compact set $K \subset \mathbb{R}^d$ and a sequence $(h_n)_{n \geq 0} \subset H$ such that:

$$\forall n, \text{ Supp } h_n \subset K \quad \text{and} \quad \lim_{n \to \infty} h_n = \varphi \quad \text{uniformly.}$$

Consequently, $t \mapsto \mu_t$ is vaguely continuous at any $s \not\in \Delta$, and, since measures $\mu_t$ are probabilities, $t \mapsto \mu_t$ is also weakly continuous at any $s \not\in \Delta$.

We may write $\Delta = \{d_j ; j \in \mathbb{N}\}$. For $n \in \mathbb{N}$, we denote by $(k_i^{(n)}, l \geq 0)$ the increasing rearrangement of the set:

$$\{k \cdot 2^{-n} ; k \in \mathbb{N}\} \cup \{d_j ; 0 \leq j \leq n\}.$$

We define $(\mu_t^{(n)}, t \geq 0)$ by:

$$\mu_t^{(n)} = \mu_{k_i^{(n)}}$$

if there exists $l$ such that $t = k_i^{(n)}$,

and by:

$$\mu_t^{(n)} = \frac{k_{i+1}^{(n)} - t}{k_{i+1}^{(n)} - k_i^{(n)}} \mu_{k_i^{(n)}} + \frac{t - k_i^{(n)}}{k_{i+1}^{(n)} - k_i^{(n)}} \mu_{k_{i+1}^{(n)}}$$

if $t \in [k_i^{(n)}, k_{i+1}^{(n)}]$.

**Lemma 4.2** The following properties hold:

1. For every $n \geq 0$, $(\mu_t^{(n)}, t \geq 0)$ is a peacock and the map: $t \mapsto \mu_t^{(n)} \in \mathcal{M}$ is continuous.

2. For any $t \geq 0$, $\sup\{\mu_t^{(n)}(|x|) ; n \in \mathbb{N}\} < \infty$.

3. For any $t \geq 0$, the set $\{\mu_t^{(n)} ; n \in \mathbb{N}\}$ is uniformly integrable.

4. For $t \geq 0$, $\lim_{n \to \infty} \mu_t^{(n)} = \mu_t \quad \text{in } \mathcal{M}$.

**Proof** Properties 1 and 4 are clear by construction. Property 2 (resp. property 3) follows directly from property 2 (resp. property 3) in Proposition 2.2.

By Theorem 3.2, there exists, for each $n$, a right-continuous martingale
\((M_t^{(n)}, t \geq 0)\) which is associated to \((\mu_t^{(n)}, t \geq 0)\). For any \(r \in \mathbb{N}\) and \(\tau_r = (t_1, \cdots, t_r) \in \mathbb{R}_+^r\), we denote by \(\Pi_{\tau_r}^{(r,n)}\) the law of \((M_t^{(n)}, \cdots, M_t^{(n)})\), a probability measure on \((\mathbb{R}^d)^r\).

**Lemma 4.3** For every \(\tau_r \in \mathbb{R}_+^r\), the set of probability measures: \(\{\Pi_{\tau_r}^{(r,n)}; n \in \mathbb{N}\}\) is tight.

**Proof** As in Lemma 3.1, for \(p > 0\),

\[
\Pi_{\tau_r}^{(r,n)}(|x|_r \geq p) \leq \frac{1}{p} \sum_{j=1}^{r} \mu_j^{(n)}(|x|),
\]

and by property 2 in Lemma 4.2,

\[
\lim_{p \to \infty} \sup_{n \geq 0} \Pi_{\tau_r}^{(r,n)}(|x|, |x| \geq p) = 0.
\]

Let now \(U\) be an ultrafilter on \(\mathbb{N}\), which refines Fréchet’s filter. As a consequence of the previous lemma, for every \(r \in \mathbb{N}\) and every \(\tau_r \in \mathbb{R}_+^r\), \(\lim_U \Pi_{\tau_r}^{(r,n)}\) exists for the weak convergence and we denote this limit by \(\Pi_{\tau_r}^{(r)}\).

By property 4 in Lemma 4.2, \(\Pi_{\tau_r}^{(1)} = \mu_t\). There exists a process \((M_t, t \geq 0)\) such that, for every \(r \in \mathbb{N}\) and every \(\tau_r \in \mathbb{R}_+^r\), the law of \((M_{t_1}, \cdots, M_{t_r})\) is \(\Pi_{\tau_r}^{(r)}\). In particular, this process \((M_t, t \geq 0)\) is associated to \((\mu_t, t \geq 0)\).

**Lemma 4.4** The process \((M_t, t \geq 0)\) is a martingale.

**Proof** The proof is quite similar to that of Lemma 3.2, but we give the details for the sake of completeness. We recall the notation:

\[
\forall p > 0, \forall x \in \mathbb{R}^d, \varphi_p(x) = \left(1 \vee \frac{|x|}{p}\right)^{-1} x.
\]

Let \(0 \leq s_1 < s_2 < \cdots < s_r \leq s \leq t\) be elements of \(\mathbb{R}_+\), and let \(f \in C_b((\mathbb{R}^d)^r)\). We set: \(\|f\|_\infty = \sup\{|f(x)|; x \in (\mathbb{R}^d)^r|\}\). Then, for every \(n\),

\[
\mathbb{E}[f(M_{s_1}^{(n)}, \cdots, M_{s_r}^{(n)}) M_t^{(n)}] = \mathbb{E}[f(M_{s_1}^{(n)}, \cdots, M_{s_r}^{(n)}) M_s^{(n)}].
\]

On the other hand,

\[
|\mathbb{E}[f(M_{s_1}, \cdots, M_{s_r}) \varphi_p(M_t)] - \mathbb{E}[f(M_{s_1}, \cdots, M_{s_r}) M_t]| \leq \|f\|_{\infty} \mu_t \left(|x| 1_{(|x| \geq p)}\right), \quad \text{for every } p > 0,
\]

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\[
\begin{align*}
& \left| \mathbb{E}[f(M_{s_1}^{(n)}, \ldots, M_{s_r}^{(n)}) \varphi_p(M_t^{(n)})] - \mathbb{E}[f(M_{s_1}^{(n)}, \ldots, M_{s_r}^{(n)}) M_t^{(n)}] \right| \\
\leq & \|f\|_\infty \mu_{t}^{(n)} \left( |x| \mathbb{1}_{\{|x| \geq p\}} \right) , \text{ for every } n \text{ and every } p > 0,
\end{align*}
\]
and likewise, replacing \( t \) by \( s \). Moreover,
\[
\lim_{U} \mathbb{E}[f(M_{s_1}^{(n)}, \ldots, M_{s_r}^{(n)}) \varphi_p(M_t^{(n)})] = \mathbb{E}[f(M_{s_1}, \ldots, M_{s_r}) \varphi_p(M_t)] ,
\]
and likewise, replacing \( t \) by \( s \). Finally, we obtain, for \( p > 0 \),
\[
\begin{align*}
& |\mathbb{E}[f(X_{s_1}, \ldots, X_{s_r}) X_t] - \mathbb{E}[f(X_{s_1}, \ldots, X_{s_r}) X_s]| \\
\leq & \ 2 \|f\|_\infty \sup_{n \geq 0} \left[ \mu_{t}^{(n)} \left( |x| \mathbb{1}_{\{|x| \geq p\}} \right) + \mu_{s}^{(n)} \left( |x| \mathbb{1}_{\{|x| \geq p\}} \right) \right] ,
\end{align*}
\]
and, by property 3 in Lemma 4.2, the desired result follows, letting \( p \) go to \( \infty \).

This lemma completes the proof of Theorem 4.1.

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References


