A new proof of Kellerer's theorem

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Abstract: In this paper, we present a new proof of the celebrated theorem of H. Kellerer, stating that every integrable process, which increases in the convex order, has the same one-dimensional marginals as a martingale. Our proof proceeds by approximations, and calls upon martingales constructed as solutions of stochastic differential equations. It relies on a uniqueness result, due to M. Pierre, for a Fokker-Planck equation.
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1 Introduction

1.1

First we fix the terminology.

We say that two $\mathbb{R}$-valued processes are *associated*, if they have the same one-dimensional marginals. A process which is associated with a martingale is called a 1-*martingale*.

An $\mathbb{R}$-valued process $(X_t\ ,\ t \geq 0)$ is called a *peacock* (see [HPRY] for the origin of this term and many examples) if:

i) it is *integrable*, that is:

$$\forall t \geq 0, \quad \mathbb{E}[|X_t|] < \infty$$

ii) it *increases in the convex order*, meaning that, for every convex function $\psi : \mathbb{R} \longrightarrow \mathbb{R}$, the map:

$$t \geq 0 \longrightarrow \mathbb{E}[\psi(X_t)] \in (-\infty, +\infty]$$

is increasing.

Actually, it may be noted that, in the definition of a peacock, only the family $(\mu_t \ , \ t \geq 0)$ of its one-dimensional marginals is involved. In the following, we shall also call a *peacock*, a family $(\mu_t \ , \ t \geq 0)$ of probability measures on $\mathbb{R}$ such that:

i) $\quad \forall t \geq 0, \quad \displaystyle\int |x| \ \mu_t(\mathrm{d}x) < \infty \ ,$

ii) $\quad$ for every convex function $\psi : \mathbb{R} \longrightarrow \mathbb{R}$, the map:

$$t \geq 0 \longrightarrow \int \psi(x) \ \mu_t(\mathrm{d}x) \in (-\infty, +\infty]$$

is increasing.

Likewise, a family $(\mu_t \ , \ t \geq 0)$ of probability measures on $\mathbb{R}$ and an $\mathbb{R}$-valued process $(Y_t \ , \ t \geq 0)$ will be said to be *associated* if, for every $t \geq 0$, the law of Y_t is μ_t, i.e. if $(\mu_t \ , \ t \geq 0)$ is the family of the one-dimensional marginals of $(Y_t \ , \ t \geq 0)$.

1.2

It is an easy consequence of Jensen's inequality that an $\mathbb{R}$-valued process $(X_t \ , \ t \geq 0)$ which is a 1-martingale, is a peacock. A remarkable result due to H. Kellerer ([K], 1972) states that, conversely, any $\mathbb{R}$-valued process $(X_t \ , \ t \geq 0)$ which is a peacock, is a 1-martingale. More precisely, Kellerer's result states that any peacock admits an associated martingale which is *Markovian*.

Recently, G. Lowther ([L], 2008) stated that if $(\mu_t \ , \ t \geq 0)$ is a peacock such that the map: $t \longrightarrow \mu_t$ is weakly continuous (i.e. for any $\mathbb{R}$-valued, bounded and continuous function f on $\mathbb{R}$, the map: $t \longrightarrow \int f(x) \ \mu_t(\mathrm{d}x)$ is continuous), then $(\mu_t \ , \ t \geq 0)$ is associated with a strongly Markovian martingale which moreover is "almost-continuous" (see [L] for the definition).

1.3

In this paper, our aim is to present a new proof of the above mentioned theorem of H. Kellerer, which eventually identifies peacocks and 1-martingales. Our method is inspired from the "Fokker-Planck Equation Method" ([HPRY, Section 6.2]) and appears then as a new application of M. Pierre's uniqueness theorem for a Fokker-Planck equation ([HPRY, Theorem 6.1]).

1.4

The remainder of this paper is organised as follows:

- In Section 2, we define as usual the *call function* C_μ of the law μ of an integrable random variable X, by:

  $$\forall x \in \mathbb{R}, \quad C_\mu(x) = \int (y-x)^+ \, \mu(\mathrm{d}y) = \mathbb{E}[(X-x)^+]$$

  and we present some properties of the correspondence: $\mu \longrightarrow C_\mu$, which are useful in the study of peacocks.

- In Section 3, we prove that a family $(\mu_t \ , \ t \geq 0)$ of probability measures on $\mathbb{R}$, is associated to a *right-continuous* martingale, if and only if, $(\mu_t \ , \ t \geq 0)$ is a peacock such that the map: $t \longrightarrow \mu_t$ is *weakly right-continuous* on $\mathbb{R}_+$.

- In Section 4, by approximation from the previous result, we deduce Kellerer's theorem in the general case.

2 Call functions and peacocks

In this section, we fix the notation and the terminology, and we gather some preliminary results.

2.1 Call functions

In the sequel, we denote by $\mathcal{M}$ the set of probability measures on $\mathbb{R}$, equipped with the topology of weak convergence (with respect to the space of $\mathbb{R}$-valued, bounded, continuous functions on $\mathbb{R}$).

We denote by $\mathcal{M}_f$ the subset of $\mathcal{M}$ consisting of measures $\mu \in \mathcal{M}$ such that $\int |x| \ \mu(\mathrm{d}x) < \infty$. $\mathcal{M}_f$ is also equipped with the topology of weak convergence.

We define, for $\mu \in \mathcal{M}_f$, the *call function* C_μ by:

$$\forall x \in \mathbb{R}, \quad C_\mu(x) = \int (y-x)^+ \; \mu(\mathrm{d}y) \; .$$

Proposition 2.1 *If $\mu \in \mathcal{M}_f$, then C_μ satisfies the following properties:*

a) C_μ *is a convex, nonnegative function on* $\mathbb{R}$.

b) $\lim_{x\to+\infty} C_\mu(x) = 0$.

c) *There exists $a \in \mathbb{R}$ such that* $\lim_{x\to-\infty}(C_\mu(x)+x) = a$.

Conversely, if a function C satisfies the above three properties, then there exists a unique $\mu \in \mathcal{M}_f$ such that $C = C_\mu$. This measure μ is the second derivative, in the sense of distributions, of the function C.

Proof Clearly, if $\mu \in \mathcal{M}_f$, then C_μ satisfies properties a), b) and c). For example, c) follows directly from:

$$\forall x \in \mathbb{R}, \quad C_\mu(x) + x = \int \sup(y,x) \; \mu(\mathrm{d}y)$$

which tends to $a = \int y \; \mu(\mathrm{d}y)$ as $x \to -\infty$. Moreover, it is easy to see that the measure μ is the second derivative, in the sense of distributions, of the function C_μ.

Conversely, let C be a function satisfying properties a), b) and c). We define μ as the second derivative, in the sense of distributions, of the function C. Then μ is a positive measure. Denote by $C'(x)$ the right derivative, at x, of the convex function C. By properties a) and b),

$$\forall x \in \mathbb{R}, \;\; C'(x) \leq 0 \quad \text{and} \quad \lim_{x\to+\infty} C'(x) = 0 \; .$$

Therefore, for $x \in \mathbb{R}$,

$$C'(x) = -\int 1_{(x,+\infty)}(y) \; \mu(\mathrm{d}y) \; .$$

By property c), $\lim_{x\to-\infty} C'(x) = -1$ and then $\mu \in \mathcal{M}$.
Besides,

$$C(x) = -\int_x^{+\infty} C'(y) \; (\mathrm{d}y) = \int (y-x)^+ \; \mu(\mathrm{d}y)$$

and

$$C(x) + x = \int \sup(y,x) \; \mu(\mathrm{d}y) \; .$$

Using again property c), we see that $\mu \in \mathcal{M}_f$ and $C = C_\mu$.

$\square$

Proposition 2.2 *Let $\mu \in \mathcal{M}_f$ and set $\mathbb{E}[\mu] = \int x \, \mu(\mathrm{d}x)$. Then C_μ satisfies the following additional properties:*

i) $\quad \forall x \leq y, \qquad 0 \leq C_\mu(x) - C_\mu(y) \leq y - x \; .$

ii) $$\forall x, \qquad C_\mu(x) + x - \mathbb{E}[\mu] = \int (x-y)^+ \, \mu(\mathrm{d}y) \; .$$

iii) $$\lim_{x \to -\infty} (C_\mu(x) + x) = \mathbb{E}[\mu] \; .$$

Proof The proposition follows from the following equalities, already seen in the previous proof:

$$C'_\mu(x) = -\int 1_{(x,+\infty)}(y) \, \mu(\mathrm{d}y) \; ,$$

$$C_\mu(x) + x = \int \sup(y,x) \, \mu(\mathrm{d}y) \; .$$

$\square$

To state the next proposition, we now recall that a subset $\mathcal{H}$ of $\mathcal{M}$ is said to be *uniformly integrable* if

$$\lim_{c \to +\infty} \sup_{\mu \in \mathcal{H}} \int_{\{|x| \geq c\}} |x| \, \mu(\mathrm{d}x) = 0 \; .$$

We remark that, if $\mathcal{H}$ is uniformly integrable, then

$$\mathcal{H} \subset \mathcal{M}_f \quad \text{and} \quad \sup\{\int |x| \, \mu(\mathrm{d}x) \; ; \; \mu \in \mathcal{H}\} < \infty \; .$$

Proposition 2.3 *Let I be a set and let $\mathcal{E}$ be a filter on I. Consider a uniformly integrable family $(\mu_i \, , \; i \in I)$ in $\mathcal{M}$, and $\mu \in \mathcal{M}$. The following properties are equivalent:*

1) $\quad \lim_{\mathcal{E}} \mu_i = \mu$ *with respect to the topology on* $\mathcal{M}$ *.*

2) $\quad \mu \in \mathcal{M}_f$ *and*

$$\forall x \in \mathbb{R}, \quad \lim_{\mathcal{E}} C_{\mu_i}(x) = C_\mu(x) \ .$$

3) $\quad \mu \in \mathcal{M}_f$ *and, for every* $\mathbb{R}$*-valued continuous function* f *on* $\mathbb{R}$ *such that*

$$\exists a > 0, b > 0, \ \ \forall x \in \mathbb{R}, \qquad |f(x)| \leq a + b\,|x| \ ,$$

one has:

$$\lim_{\mathcal{E}} \int f(x)\ \mu_i(\mathrm{d}x) = \int f(x)\ \mu(\mathrm{d}x) \ .$$

Proof We first assume that property 1) holds. Then

$$\int |x|\ \mu(\mathrm{d}x) \leq \sup\{\int |x|\ \mu_i(\mathrm{d}x)\ ;\ i \in I\} < \infty \ ,$$

and $\mu \in \mathcal{M}_f$. Let f be an $\mathbb{R}$-valued continuous function on $\mathbb{R}$ such that

$$\exists a > 0, b > 0, \ \ \forall x \in \mathbb{R}, \qquad |f(x)| \leq a + b\,|x| \ .$$

We set, for $n \in \mathbb{N}$, $f_n(x) = \sup[\inf(f, n), -n]$. Since f_n is continuous and bounded,

$$\lim_{\mathcal{E}} \int f_n(x)\ \mu_i(\mathrm{d}x) = \int f_n(x)\ \mu(\mathrm{d}x) \ .$$

On the other hand, for $n \geq a$,

$$|f(x) - f_n(x)| = (|f(x)| - n)^+ \leq (b\,|x| + a - n)^+ \leq b\,|x|\,1_{\{|x| \geq \frac{n-a}{b}\}} \ ,$$

and hence

$$\sup_{i \in I} \left| \int f(x)\ \mu_i(\mathrm{d}x) - \int f_n(x)\ \mu_i(\mathrm{d}x) \right| \leq b \sup_{i \in I} \int_{\{|x| \geq \frac{n-a}{b}\}} |x|\ \mu_i(\mathrm{d}x) \ .$$

By uniform integrability, we then obtain:

$$\lim_{n \to \infty} \sup_{i \in I} \left| \int f(x)\ \mu_i(\mathrm{d}x) - \int f_n(x)\ \mu_i(\mathrm{d}x) \right| = 0 \ .$$

Finally,

$$\begin{aligned} \int f(x)\ \mu(\mathrm{d}x) &= \lim_{n \to \infty} \lim_{\mathcal{E}} \int f_n(x)\ \mu_i(\mathrm{d}x) \\ &= \lim_{\mathcal{E}} \lim_{n \to \infty} \int f_n(x)\ \mu_i(\mathrm{d}x) = \lim_{\mathcal{E}} \int f(x)\ \mu_i(\mathrm{d}x) \ , \end{aligned}$$

and property 3) is satisfied.

Obviously, property 3) entails property 2).

Suppose then that property 2) holds. By equicontinuity (property i) in Proposition 2.2),

$$\lim_{\mathcal{E}} C_{\mu_i}(x) = C_\mu(x)$$

uniformly on compact sets of $\mathbb{R}$, and hence in the sense of distributions. Consequently, since μ_i (resp. μ) is the second derivative, in the sense of distributions, of the function C_{μ_i} (resp. C_μ),

$$\lim_{\mathcal{E}} \mu_i = \mu$$

in the sense of distributions. As μ_i and μ are probability measures, this entails property 1). □

2.2 Peacocks

In this subsection, we fix a family $(\mu_t \ , \ t \geq 0)$ in $\mathcal{M}_f$ and we define a function $C(t,x)$ on $\mathbb{R}_+ \times \mathbb{R}$ by:

$$C(t,x) = C_{\mu_t}(x) \ .$$

We recall (see Subsection 1.1) that the family $(\mu_t \ , \ t \geq 0)$ is called a *peacock*, if

i) $\forall t \geq 0, \quad \int |x| \ \mu_t(\mathrm{d}x) < \infty \ ,$

ii) for every convex function $\psi : \mathbb{R} \longrightarrow \mathbb{R}$, the map:

$$t \geq 0 \longrightarrow \int \psi(x) \ \mu_t(\mathrm{d}x) \in (-\infty, +\infty]$$

is increasing.

The following characterization is easy to prove and is stated in [HPRY, Exercise 1.7].

Proposition 2.4 *The family $(\mu_t \ , \ t \geq 0)$ is a peacock if and only if:*

1. *the expectation $\mathbb{E}[\mu_t]$ does not depend on t,*
2. *for every $x \in \mathbb{R}$, the function $t \geq 0 \longrightarrow C(t,x)$ is increasing.*

The following proposition plays an important role in the sequel.

Proposition 2.5 *Assume that* $(\mu_t\ ,\ t \geq 0)$ *is a peacock, and let* $T > 0$. *Then,*

(1) *the set* $\{\mu_t\ ;\ 0 \leq t \leq T\}$ *is uniformly integrable,*

(2) $\lim\limits_{|x|\to\infty} \sup\{C(t,x) - C(s,x)\ ;\ 0 \leq s \leq t \leq T\} = 0$.

Proof Property (1) is stated in [HPRY, Exercise 1.1]. Actually, it suffices to remark that, if $c \geq 0$,

$$|x|\, 1_{\{|x|\geq c\}} \leq (2\,|x| - c)^+ \ .$$

As the function $x \longrightarrow (2\,|x| - c)^+$ is convex,

$$\sup_{t\in[0,T]} \int_{\{|x|\geq c\}} |x|\ \mu_t(\mathrm{d}x) \leq \int (2\,|x| - c)^+\ \mu_T(\mathrm{d}x)$$

Now, by dominated convergence,

$$\lim_{c\to+\infty} \int (2\,|x| - c)^+\ \mu_T(\mathrm{d}x) = 0\ .$$

We have:

$$\sup\{C(t,x) - C(s,x)\ ;\ 0 \leq s \leq t \leq T\} \leq C(T,x)\ .$$

Hence, by property b) in Proposition 2.1,

$$\lim_{x\to+\infty} \sup\{C(t,x) - C(s,x)\ ;\ 0 \leq s \leq t \leq T\} = 0\ .$$

On the other hand, since $\mathbb{E}[\mu_t]$ does not depend on t,

$$C(t,x) - C(s,x) = [C(t,x) + x - \mathbb{E}[\mu_t]] - [C(s,x) + x - \mathbb{E}[\mu_s]]$$

Now, by property ii) in Proposition 2.2,

$$C(t,x) + x - \mathbb{E}[\mu_t] = \int (x - y)^+\ \mu_t(\mathrm{d}y)\ ,$$

is therefore nonnegative and increases with respect to t. Hence

$$\sup\{C(t,x) - C(s,x)\ ;\ 0 \leq s \leq t \leq T\} \leq C(T,x) + x - \mathbb{E}[\mu_T]$$

and, by property iii) in Proposition 2.2,

$$\lim_{x\to-\infty} \sup\{C(t,x) - C(s,x)\ ;\ 0 \leq s \leq t \leq T\} = 0\ .$$

□

3 Right-continuous peacoks

In this section, we shall prove Kellerer's theorem for right-continuous peacoks. We proceed by regularization, using, for regularized peacocks, the Fokker-Planck equation method as in [HPRY, Chapter 6]. This method relies heavily on M. Pierre's uniqueness theorem for a Fokker-Planck equation ([HPRY, Theorem 6.1]).

We first recall the main result in the Fokker-Planck equation method, namely Theorem 6.2 in [HPRY]. The next statement is a slightly extended version of this theorem.

Theorem 3.1 (see Theorem 6.2 in [HPRY]) *Let* $U = (0, +\infty) \times \mathbb{R}$ *and* $\overline{U}$ *the closure of* U $(\overline{U} = \mathbb{R}_+ \times \mathbb{R})$*. Let* σ *be a continuous function on* $\overline{U}$ *such that* $\sigma(t,x) > 0$ *for every* $(t,x) \in U$*. Let* $\mu \in \mathcal{M}_f$*.*

1) The stochastic differential equation

$$Z_t = Z_0 + \int_0^t \sigma(s, Z_s)\,\mathrm{d}B_s$$

(where Z_0 *is a random variable with law* μ*, independent of the Brownian motion* $(B_s, s \geq 0)$*) admits a weak non-exploding solution* $(Y_t\ ,\ t \geq 0)$ *, which is unique in law.*

2) Let $p(t, \mathrm{d}x)$ *be the law of* Y_t*. Then,* $(p(t, \mathrm{d}x)\ ,\ t \geq 0)$ *is the* unique *family in* $\mathcal{M}$ *such that:*

$$t \longrightarrow p(t, \mathrm{d}x) \in \mathcal{M} \quad \textit{is continuous and} \quad p(0, \mathrm{d}x) = \mu(\mathrm{d}x)\ ,$$

$$\frac{\partial p}{\partial t} - \frac{1}{2}\frac{\partial^2}{\partial x^2}(\sigma^2\, p) = 0 \quad \textit{in the sense of distributions on } U\ .$$

We now present our proof of Kellerer's theorem for right-continuous peacoks.

Theorem 3.2 *Let* $(\mu_t\ ,\ t \geq 0)$ *be a family in* $\mathcal{M}$*. Then the following properties are equivalent:*

1) There exists a right-continuous martingale associated to $(\mu_t\ ,\ t \geq 0)$ *.*

2) $(\mu_t\ ,\ t \geq 0)$ *is a peacock and the map:*

$$t \geq 0 \longrightarrow \mu_t \in \mathcal{M}$$

is right-continuous.

Proof We first assume that property 1) is satisfied. Then, the fact that $(\mu_t\ ,\ t \geq 0)$ is a peacock follows classically from Jensen's inequality. Let $(M_t\ ,\ t \geq 0)$ be a right-continuous martingale associated to $(\mu_t\ ,\ t \geq 0)$. Then, if f is a bounded continuous function, we obtain by dominated convergence that, for any $t \geq 0$,

$$\lim_{s \to t, s>t} \int f(x)\, \mu_s(\mathrm{d}x) = \lim_{s \to t, s>t} \mathbb{E}[f(M_s)] = \mathbb{E}[f(M_t)] = \int f(x)\, \mu_t(\mathrm{d}x)\ .$$

Therefore, the map:

$$t \geq 0 \longrightarrow \mu_t \in \mathcal{M}$$

is right-continuous, and property 2) is satisfied.

Conversely, we now assume that property 2) is satisfied. We set, as in Subsection 2.2, $C(t,x) = C_{\mu_t}(x)$. Let α be a density of probability on $\mathbb{R}$, of C^∞ class , with compact support contained in $[0,1]$. We set, for $\varepsilon \in (0,1)$ and $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$p_\varepsilon(t,x) = \frac{1-\varepsilon}{\varepsilon} \int \alpha(u) \left[\int \alpha\left(\frac{y-x}{\varepsilon}\right)\ \mu_{t+\varepsilon u}(\mathrm{d}y) \right]\ \mathrm{d}u + \varepsilon\, g(t,x)$$

with

$$g(t,x) = \frac{1}{\sqrt{2\,\pi\,(1+t)}}\ \exp\left(-\frac{x^2}{2\,(1+t)}\right)\ .$$

Lemma 3.1 *The function* p_ε *is of* C^∞ *class on* $\mathbb{R}_+ \times \mathbb{R}$ *and* $p_\varepsilon(t,x) > 0$ *for any* (t,x). *Moreover,*

$$\int p_\varepsilon(t,x)\, \mathrm{d}x = 1 \quad \textit{and} \quad \int |x|\, p_\varepsilon(t,x)\, \mathrm{d}x < \infty\ .$$

The proof is straightforward.
We now set:

$$\mu_t^\varepsilon(\mathrm{d}x) = p_\varepsilon(t,x)\, \mathrm{d}x\ .$$

By Lemma 3.1, $\mu_t^\varepsilon \in \mathcal{M}_f$ and we set:

$$C_\varepsilon(t,x) = C_{\mu_t^\varepsilon}(x)\ .$$

Lemma 3.2 *For any* $t \geq 0$, *the set* $\{\mu_t^\varepsilon\ ;\ 0 < \varepsilon < 1\}$ *is uniformly integrable.*

Proof Let $a = \int y\,\alpha(y)\,\mathrm{d}y$. A simple computation yields:

$$\begin{aligned}\int_{\{|x|\geq c\}} |x|\;\mu_t^\varepsilon(\mathrm{d}x) &\leq \int \alpha(u)\left[\int_{\{|y|\geq c-1\}} (|y|+a)\;\mu_{t+\varepsilon u}(\mathrm{d}y)\right]\,\mathrm{d}u \\ &\quad + \int_{\{|x|\geq c\}} |x|\,g(t,x)\,\mathrm{d}x\end{aligned}$$

and the result follows from the uniform integrability of $\{\mu_v\;;\;0\leq v\leq t+1\}$ (property (1) in Proposition 2.5).

$\square$

Lemma 3.3 *One has:*

$$\begin{aligned}C_\varepsilon(t,x) &= (1-\varepsilon)\int\int \alpha(u)\,\alpha(y)\,C(t+\varepsilon u, x+\varepsilon y)\,\mathrm{d}y\,\mathrm{d}u \\ &\quad +\varepsilon\int_x^{+\infty} (y-x)\,g(t,y)\,\mathrm{d}y\ .\end{aligned}$$

The function C_ε *is of* C^∞ *class on* $\mathbb{R}_+\times\mathbb{R}$. *Moreover, for any* $(t,x)\in\mathbb{R}_+\times\mathbb{R}$,

$$\frac{\partial C_\varepsilon}{\partial t}(t,x) > 0 \qquad \textit{and} \qquad \frac{\partial^2 C_\varepsilon}{\partial x^2}(t,x) = p_\varepsilon(t,x)\ .$$

Proof The above expression of C_ε follows directly from the definitions. We deduce therefrom that C_ε is of C^∞ class on $\mathbb{R}_+\times\mathbb{R}$. Now, by property 2. in Proposition 2.4,

$$\frac{\partial C_\varepsilon}{\partial t}(t,x) \geq \varepsilon\,\frac{\partial}{\partial t}\left[\int_x^{+\infty} (y-x)\,g(t,y)\,\mathrm{d}y\right] = \frac{\varepsilon}{2}\,g(t,x) > 0\ .$$

Finally, the equality:

$$\frac{\partial^2 C_\varepsilon}{\partial x^2}(t,x) = p_\varepsilon(t,x)$$

holds, since, by Proposition 2.1, it holds in the sense of distributions, and both sides are continuous.

$\square$

Lemma 3.4 *For* $0\leq s\leq t$,

$$\lim_{|x|\to\infty}\sup\{C_\varepsilon(t,x)-C_\varepsilon(s,x)\;;\;0<\varepsilon<1\} = 0\ .$$

Proof By Lemma 3.3,

$$\sup\{C_\varepsilon(t,x) - C_\varepsilon(s,x)\ ;\ 0<\varepsilon<1\} \le A(x) + B(x)$$

with

$$A(x) = \sup\{C(w,y) - C(v,y)\ ;\ 0 \le v \le w \le t+1, x \le y \le x+1\}$$

and

$$B(x) = \frac{1}{2}\int_s^t g(u,x)\,\mathrm{d}u\ .$$

By property (2) in Proposition 2.5, $\lim_{|x|\to\infty} A(x) = 0$, and, obviously, $\lim_{|x|\to\infty} B(x) = 0$.

□

Lemma 3.5 *For* $t \ge 0$,

$$\lim_{\varepsilon\to 0} \mu_t^\varepsilon = \mu_t \qquad \textit{in } \mathcal{M}\ .$$

Proof By property i) in Proposition2.2, property (1) in Proposition 2.5 and Proposition 2.3,

$$\lim_{s\to t, s>t} C(s,x) = C(t,x) \qquad \text{uniformly on compact sets.}$$

Then, the expression of C_ε in Lemma 3.3 yields:

$$\lim_{s\to t, s>t} C_\varepsilon(s,x) = C_\varepsilon(t,x)\ .$$

It then suffices to apply again Proposition 2.3, taking into account Lemma 3.2.
Note that we might also have proven this lemma directly from the definition of μ_t^ε.

□

Lemma 3.6 *We set, for* $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$\sigma_\varepsilon(t,x) = \left(2\,\frac{\frac{\partial C_\varepsilon}{\partial t}(t,x)}{p_\varepsilon(t,x)} \right)^{1/2}\ .$$

Then, σ_ε is continuous and strictly positive on $\mathbb{R}_+ \times \mathbb{R}$. Moreover, for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$\frac{\partial p_\varepsilon}{\partial t}(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\sigma_\varepsilon^2(t,x)\, p_\varepsilon(t,x)\right) \ ,$$

which is the Fokker-Planck equation for p_ε.

Proof This is a direct consequence of Lemma 3.1 and Lemma 3.3. In particular, the Fokker-Planck equation can be written:

$$\frac{\partial}{\partial t}\frac{\partial^2 C_\varepsilon}{\partial x^2} = \frac{\partial^2}{\partial x^2}\frac{\partial C_\varepsilon}{\partial t} \ .$$

□

By Theorem 3.1, there exists a process $(M_t^\varepsilon \ , \ t \geq 0)$ which is a weak solution of the stochastic differential equation

$$Z_t = Z_0 + \int_0^t \sigma_\varepsilon(s, Z_s)\, \mathrm{d}B_s$$

with Z_0 a random variable with law μ_0^ε, independent of the Brownian motion $(B_s, s \geq 0)$, and this process $(M_t^\varepsilon \ , \ t \geq 0)$ is associated to $(\mu_t^\varepsilon \ , \ t \geq 0)$. For every $n \in \mathbb{N}$ and $\tau_n = (t_1, \cdots, t_n) \in \mathbb{R}_+^n$, we denote by $\mu_{\tau_n}^{(\varepsilon,n)}$ the law of $(M_{t_1}^\varepsilon, \cdots, M_{t_n}^\varepsilon)$, a probability on $\mathbb{R}^n$.

Lemma 3.7 *For every $n \in \mathbb{N}$ and $\tau_n \in \mathbb{R}_+^n$, the set of probability measures: $\{\mu_{\tau_n}^{(\varepsilon,n)} \ ; \ 0 < \varepsilon < 1\}$, is tight.*

Proof Let $n \in \mathbb{N}$ and $\tau_n = (t_1, \cdots, t_n) \in \mathbb{R}_+^n$. For $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, we set $|x| := \sup_{1 \leq j \leq n} |x_j|$. Then, for $c > 0$,

$$\begin{aligned}
\mu_{\tau_n}^{(\varepsilon,n)}(|x| \geq c) &= \mathbb{P}\left(\sup_{1 \leq j \leq n} |M_{t_j}^\varepsilon| \geq c\right) \leq \frac{1}{c}\,\mathbb{E}\left[\sup_{1 \leq j \leq n} |M_{t_j}^\varepsilon|\right] \\
&\leq \frac{1}{c}\sum_{j=1}^n \mathbb{E}\left[|M_{t_j}^\varepsilon|\right] \leq \frac{1}{c}\sum_{j=1}^n \int |x|\ \mu_{t_j}^\varepsilon(\mathrm{d}x) \ .
\end{aligned}$$

Now, by Lemma 3.2, for $1 \leq j \leq n$,

$$\sup_{0<\varepsilon<1} \int |x|\ \mu_{t_j}^\varepsilon(\mathrm{d}x) < \infty \ .$$

Thus,

$$\lim_{c \to +\infty} \sup_{0<\varepsilon<1} \mu_{\tau_n}^{(\varepsilon,n)}(|x| \geq c) = 0 \ ,$$

which yields the tightness of $\{\mu^{(\varepsilon,n)}_{\tau_n}\ ;\ 0<\varepsilon<1\}$.

□

As a consequence of the previous lemma, and with the help of the diagonal procedure, there exists a sequence $(\varepsilon_p\ ,\ p\geq 0)$ tending to 0 such that, for every $n\in\mathbb{N}$ and every $\tau_n\in\mathbb{Q}_+^n$, the sequence of probabilities on $\mathbb{R}^n$: $(\mu^{(\varepsilon_p,n)}_{\tau_n}\ ,\ p\geq 0)$, weakly converges to a probability which we denote by $\mu^n_{\tau_n}$. We remark that, by Lemma 3.5, for any $t\in\mathbb{Q}_+$, $\mu^1_t=\mu_t$. There exists a process $(M_t\ ,\ t\in\mathbb{Q}_+)$ such that, for every $n\in\mathbb{N}$ and every $\tau_n=(t_1,\cdots,t_n)\in\mathbb{Q}_+^n$, the law of $(M_{t_1},\cdots,M_{t_n})$ is $\mu^n_{\tau_n}$.

Lemma 3.8 *The process* $(M_t\ ,\ t\in\mathbb{Q}_+)$ *is a martingale.*

Proof Let ϕ be a C^2-function on $\mathbb{R}$ such that $\phi(x)=1$ for $|x|\leq 1$, $\phi(x)=0$ for $|x|\geq 2$, and $0\leq\phi(x)\leq 1$ for all $x\in\mathbb{R}$. We set, for $k>0$, $\phi_k(x)=x\,\phi(k^{-1}x)$. Fix now $n\in\mathbb{N}$ and n continuous bounded functions $(g_1,\cdots,g_n)$ on $\mathbb{R}$, and finally $0\leq s_1\leq\cdots\leq s_n\leq s\leq t$ elements of $\mathbb{Q}_+$. We set:

$$\begin{aligned}\Theta(p,k) &= \mathbb{E}[g_1(M^{\varepsilon_p}_{s_1})g_2(M^{\varepsilon_p}_{s_2})\cdots g_n(M^{\varepsilon_p}_{s_n})\,\phi_k(M^{\varepsilon_p}_t)]\\ &\quad -\mathbb{E}[g_1(M^{\varepsilon_p}_{s_1})g_2(M^{\varepsilon_p}_{s_2})\cdots g_n(M^{\varepsilon_p}_{s_n})\,\phi_k(M^{\varepsilon_p}_s)]\ .\end{aligned}$$

From the definitions, we obtain:

$$\begin{aligned}\lim_{p\to\infty}\Theta(p,k) &= \mathbb{E}[g_1(M_{s_1})g_2(M_{s_2})\cdots g_n(M_{s_n})\,\phi_k(M_t)]\\ &\quad -\mathbb{E}[g_1(M_{s_1})g_2(M_{s_2})\cdots g_n(M_{s_n})\,\phi_k(M_s)]\end{aligned}$$

and, by dominated convergence,

$$\begin{aligned}\lim_{k\to\infty}\lim_{p\to\infty}\Theta(p,k) &= \mathbb{E}[g_1(M_{s_1})g_2(M_{s_2})\cdots g_n(M_{s_n})\,M_t]\\ &\quad -\mathbb{E}[g_1(M_{s_1})g_2(M_{s_2})\cdots g_n(M_{s_n})\,M_s]\ .\end{aligned}$$

On the other hand, by Itô's formula, we have:

$$|\Theta(p,k)|\leq m\int\int_s^t|\phi''_k(x)|\,\frac{\partial C_{\varepsilon_p}}{\partial u}(u,x)\,\mathrm{d}u\,\mathrm{d}x$$

with

$$m=\prod_{j=1}^n\sup_{x\in\mathbb{R}}|g_j(x)|\ .$$

Besides,

$$\int|\phi''_k(x)|\,\mathrm{d}x=\int|x\,\phi''(x)+2\phi'(x)|\,\mathrm{d}x$$

and $\phi_k''(x) = 0$ for $|x| \notin [k, 2k]$. Therefore, there exists a constant $\tilde{m}$ such that:

(1) $\qquad |\Theta(p,k)| \leq \tilde{m} \sup\{C_{\varepsilon_p}(t,y) - C_{\varepsilon_p}(s,y) \; ; \; k \leq |y| \leq 2k\}$.

Thus, by Lemma 3.4,

$$\lim_{k\to\infty} \Theta(p,k) = 0 \qquad \text{uniformly with respect to } p.$$

Consequently,

$$\begin{aligned} 0 &= \lim_{p\to\infty} \lim_{k\to\infty} \Theta(p,k) = \lim_{k\to\infty} \lim_{p\to\infty} \Theta(p,k) \\ &= \mathbb{E}[g_1(M_{s_1})g_2(M_{s_2}) \cdots g_n(M_{s_n})\, M_t] - \mathbb{E}[g_1(M_{s_1})g_2(M_{s_2}) \cdots g_n(M_{s_n})\, M_s] \, , \end{aligned}$$

which yields the desired result.

□

By the classical theory of martingales (see, for example, [DM]), almost surely, for every $t \geq 0$,

$$\widetilde{M}_t = \lim_{s\to t, s\in\mathbb{Q}, s>t} M_s$$

is well defined, and $(\widetilde{M}_t \ , \ t \geq 0)$ is a right-continuous martingale which, obviously, is associated to $(\mu_t \ , \ t \geq 0)$.

□

Remark By considering only the parameter k, the proof of Lemma 3.8 also shows that, for every $\varepsilon \in (0,1)$, the process $(M_t^\varepsilon \ , \ t \geq 0)$ is a (continuous) martingale.

In the following lemma, which will be useful in the next section, we state a property which is satisfied by the martingale $(\widetilde{M}_t \ , \ t \geq 0)$ constructed in the proof of Theorem 3.2.

Lemma 3.9 *Let* $g_1, \cdots, g_n, \phi_k$ *and* $\tilde{m}$ *be as in the proof of Lemma 3.8. Then, for* $0 \leq s_1 \leq \cdots \leq s_n \leq s \leq t$ *elements of* $\mathbb{R}_+$,

$$\begin{aligned} &\left| \mathbb{E}[g_1(\widetilde{M}_{s_1}) \cdots g_n(\widetilde{M}_{s_n})\, \phi_k(\widetilde{M}_t)] - \mathbb{E}[g_1(\widetilde{M}_{s_1}) \cdots g_n(\widetilde{M}_{s_n})\, \phi_k(\widetilde{M}_s)] \right| \\ &\qquad \leq \tilde{m} \sup\{C(t,y) - C(s,y) \; ; \; k \leq |y| \leq 2k\} \, . \end{aligned}$$

Proof We first suppose that $0 \leq s_1 \leq \cdots \leq s_n \leq s \leq t$ are elements of $\mathbb{Q}_+$, and we keep the notation in the proof of Lemma 3.8. By Lemma 3.5, Lemma 3.2 and Proposition 2.3, for any $t \geq 0$,

$$\lim_{p\to\infty} C_{\epsilon_p}(t,x) = C(t,x) \quad \text{uniformly on compact sets.}$$

Therefore, letting p tend to ∞ in inequality (1), we get :

$$|\mathbb{E}[g_1(M_{s_1})\cdots g_n(M_{s_n})\,\phi_k(M_t)] - \mathbb{E}[g_1(M_{s_1})\cdots g_n(M_{s_n})\,\phi_k(M_s)]|$$
$$\leq \tilde{m}\ \sup\{C(t,y) - C(s,y)\ ;\ k \leq |y| \leq 2k\}\ .$$

Suppose now that $0 \leq s_1 \leq \cdots \leq s_n \leq s \leq t$ are elements of $\mathbb{R}_+$. Using again Proposition 2.3 (and property (1) in Proposition 2.5), we obtain the desired result by approximation, from the above inequality.

$\square$

4 Kellerer's theorem: the general case

We now obtain, by approximation, a proof of Kellerer's theorem in the general case.

Theorem 4.1 *Let* $(\mu_t\ ,\ t \geq 0)$ *be a family in* $\mathcal{M}$*. Then the following properties are equivalent:*

1) There exists a martingale associated to $(\mu_t\ ,\ t \geq 0)$ *.*

2) $(\mu_t\ ,\ t \geq 0)$ *is a peacock.*

Proof We consider a peacock $(\mu_t\ ,\ t \geq 0)$ and we set $C(t,x) = C_{\mu_t}(x)$.

Lemma 4.1 *There exists a countable set* $D \subset \mathbb{R}_+$ *such that the map:*

$$t \longrightarrow \mu_t \in \mathcal{M}$$

is continuous at any $s \notin D$*.*

Proof By property 2. in Proposition 2.4, there exists a countable set $D \subset \mathbb{R}_+$ such that, for every $x \in \mathbb{Q}$, the map:

$$t \longrightarrow C(t,x)$$

is continuous at any $s \notin D$. By equicontinuity (property i) in Proposition 2.2), this continuity property holds for every $x \in \mathbb{R}$. It suffices then to apply Proposition 2.3, taking into account property (1) in Proposition 2.5.

$\square$

We may write $D = \{d_n \ ; \ n \in \mathbb{N}\}$. For $p \in \mathbb{N}$, we denote by $(k_n^{(p)} \ , \ n \geq 0)$ the increasing rearrangement of the set:

$$\{k\,2^{-p} \ ; \ k \in \mathbb{N}\} \cup \{d_j \ ; \ 0 \leq j \leq p\} \ .$$

We define $(\mu_t^{(p)} \ , \ t \geq 0)$ by:

$$\mu_t^{(p)} = \mu_{k_n^{(p)}} \qquad \text{if there exists } n \text{ such that } t = k_n^{(p)},$$

$$\text{and by: } \ \mu_t^{(p)} = \frac{k_{n+1}^{(p)} - t}{k_{n+1}^{(p)} - k_n^{(p)}}\,\mu_{k_n^{(p)}} + \frac{t - k_n^{(p)}}{k_{n+1}^{(p)} - k_n^{(p)}}\,\mu_{k_{n+1}^{(p)}} \qquad \text{if } t \in [k_n^{(p)}, k_{n+1}^{(p)}].$$

We also set: $C_p(t,x) = C_{\mu_t^{(p)}}(x)$.

Lemma 4.2 *The following properties hold:*

i) $(\mu_t^{(p)} \ , \ t \geq 0)$ *is a peacock and the map:* $t \longrightarrow \mu_t^{(p)} \in \mathcal{M}$ *is continuous.*

ii) For any $t \geq 0$, *the set* $\{\mu_t^{(p)} \ ; \ p \in \mathbb{N}\}$ *is uniformly integrable.*

iii) For $t \geq 0$, $\lim_{p \to \infty} \mu_t^{(p)} = \mu_t$ *in* $\mathcal{M}$.

iv) For $0 \leq s \leq t$,

$$\lim_{|x| \to \infty} \sup\{C_p(t,x) - C_p(s,x) \ ; \ p \geq 0\} = 0 \ .$$

Proof Properties i) and iii) are clear by construction. Property ii) (resp. property iv)) follows directly from property (1) (resp. property (2)) in Proposition 2.5.

$\square$

By Theorem 3.2, there exists, for each p, a right-continuous martingale $(M_t^{(p)} \ , \ t \geq 0)$ which is associated to $(\mu_t^{(p)} \ , \ t \geq 0)$ and satisfies the property stated in Lemma 3.9. For any $n \in \mathbb{N}$ and $\tau_n = (t_1, \cdots, t_n) \in \mathbb{R}_+^n$, we denote by $\mu_{\tau_n}^{(p,n)}$ the law of $(M_{t_1}^{(p)}, \cdots, M_{t_n}^{(p)})$, a probability measure on $\mathbb{R}^n$. The proof of the following lemma is quite similar to that of Lemma 3.7, hence we omit this proof.

Lemma 4.3 *For every $n \in \mathbb{N}$ and $\tau_n \in \mathbb{R}_+^n$, the set of probability measures $\{\mu_{\tau_n}^{(p,n)} \ ; \ p \geq 0\}$, is tight.*

Let now $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$, which refines Fréchet's filter. As a consequence of the previous lemma, for every $n \in \mathbb{N}$ and every $\tau_n \in \mathbb{R}_+^n$, $\lim_{\mathcal{U}} \mu_{\tau_n}^{(p,n)}$ exists in $\mathcal{M}$ and we denote this limit by $\mu_{\tau_n}^{(\infty,n)}$. By property iii) in Lemma 4.2, $\mu_t^{(\infty,1)} = \mu_t$. There exists a process $(M_t \ , \ t \geq 0)$ such that, for every $n \in \mathbb{N}$ and every $\tau_n = (t_1, \cdots, t_n) \in \mathbb{R}_+^n$, the law of $(M_{t_1}, \cdots, M_{t_n})$ is $\mu_{\tau_n}^{(\infty,n)}$. In particular, this process $(M_t \ , \ t \geq 0)$ is associated to $(\mu_t \ , \ t \geq 0)$.

Lemma 4.4 *The process $(M_t \ , \ t \geq 0)$ is a martingale.*

Proof The proof is similar to that of Lemma 3.8, but we give the details for the sake of completeness.

Let ϕ be a C^2-function on $\mathbb{R}$ such that $\phi(x) = 1$ for $|x| \leq 1$, $\phi(x) = 0$ for $|x| \geq 2$, and $0 \leq \phi(x) \leq 1$ for all $x \in \mathbb{R}$. We set, for $k > 0$, $\phi_k(x) = x\, \phi(k^{-1} x)$. Fix now $n \in \mathbb{N}$ and n continuous bounded functions $(g_1, \cdots, g_n)$ on $\mathbb{R}$, and finally $0 \leq s_1 \leq \cdots \leq s_n \leq s \leq t$ elements of $\mathbb{R}_+$. We set:

$$\begin{aligned}\Lambda(p,k) &= \mathbb{E}[g_1(M_{s_1}^{(p)}) g_2(M_{s_2}^{(p)}) \cdots g_n(M_{s_n}^{(p)})\, \phi_k(M_t^{(p)})] \\ &\quad - \mathbb{E}[g_1(M_{s_1}^{(p)}) g_2(M_{s_2}^{(p)}) \cdots g_n(M_{s_n}^{(p)})\, \phi_k(M_s^{(p)})] \ .\end{aligned}$$

From the definitions, we obtain, for every k,

$$\begin{aligned}\lim_{\mathcal{U}} \Lambda(p,k) &= \mathbb{E}[g_1(M_{s_1}) g_2(M_{s_2}) \cdots g_n(M_{s_n})\, \phi_k(M_t)] \\ &\quad - \mathbb{E}[g_1(M_{s_1}) g_2(M_{s_2}) \cdots g_n(M_{s_n})\, \phi_k(M_s)]\end{aligned}$$

and, by dominated convergence,

$$\begin{aligned}\lim_{k\to\infty} \lim_{\mathcal{U}} \Lambda(p,k) &= \mathbb{E}[g_1(M_{s_1}) g_2(M_{s_2}) \cdots g_n(M_{s_n})\, M_t] \\ &\quad - \mathbb{E}[g_1(M_{s_1}) g_2(M_{s_2}) \cdots g_n(M_{s_n})\, M_s] \ .\end{aligned}$$

On the other hand, since $(M_t^{(p)} \ , \ t \geq 0)$ satisfies the property stated in Lemma 3.9, there exists a constant $\tilde{m}$ such that:

$$|\Lambda(p,k)| \leq \tilde{m} \sup\{C_p(t,y) - C_p(s,y) \ ; \ k \leq |y| \leq 2k\} \ .$$

Thus, by property iv) in Lemma 4.2,

$$\lim_{k\to\infty} \Lambda(p,k) = 0 \qquad \text{uniformly with respect to } p.$$

Consequently,

$$0 = \lim_{\mathcal{U}} \lim_{k\to\infty} \Lambda(p,k) = \lim_{k\to\infty} \lim_{\mathcal{U}} \Lambda(p,k)$$
$$= \mathbb{E}[g_1(M_{s_1})g_2(M_{s_2})\cdots g_n(M_{s_n})\, M_t] - \mathbb{E}[g_1(M_{s_1})g_2(M_{s_2})\cdots g_n(M_{s_n})\, M_s]\ ,$$

which yields the desired result.

□

This lemma completes the proof of Theorem4.1. □

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