Limit theorems in the Fourier transform method for the estimation of multivariate volatility

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Abstract

In this paper, we prove some limit theorems for the Fourier estimator of multivariate volatility proposed by Malliavin and Mancino (\([14]\), \([15]\)). In a general framework of discrete time observations we establish the convergence of the estimator and some associated central limit theorems with explicit asymptotic variance. In particular, our results show that this estimator is consistent for synchronous data, but possibly biased for non-synchronous observations. Moreover, from our general central limit theorem, we deduce that the estimator can be efficient in the case of a synchronous regular sampling. In the non synchronous sampling case, the expression of the asymptotic variance is in general less tractable. We study this case more precisely through the example of an alternate sampling.

Keywords: non parametric estimation, Itô process, Fourier transform, weak convergence.

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1. Introduction

A large literature is devoted to the computation of the volatility, or the integrated volatility, of financial asset returns. In practice, one observes at discrete times some asset prices which time evolution is given by a multivariate Itô process \(X\). In this typical framework of high frequency data, volatility can be estimated through parametric or non parametric methods. Many of these methods rely on the quadratic variation formula (\([10]\), \([3]\), \([1]\)). Subsequently, modifications of the quadratic variation method have been introduced in order to cope with specific difficulties arising with financial data. Presence of jumps in data leads to the use of bi-power variation instead of the quadratic variation.

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([4], [2]), microstructure noises are handled with pre-averaged variation or related methods ([22], [12], [20], [9], [21]). In the case of the estimation of cross-volatility for assets observed at non synchronous instants, some alternative methods were introduced by Hayashi and Yoshida ([6], [7]). Indeed, in case of non synchronous data, the quadratic variation method yields to a biased estimation of the cross-volatility.

In [14], Malliavin and Mancino propose an estimator of the volatility based on the computation of $N$ Fourier coefficients of an Itô process $X$ and study some of its properties in [15]. A main advantage of this method is that the estimator only relies on the Fourier coefficients of each components of $X$ computed separately, and so the method suits well for non-synchronous data. Moreover, a proxy for these Fourier coefficients seems easily available to the statistician and the method has direct applications in empirical studies ([17], [16]).

In this paper we study the asymptotic properties of the Fourier estimator of the integrated volatility and give new results. In a general framework we prove the convergence and the asymptotic normality of the estimator based on discrete observation of the process $X$, possibly irregular and non synchronous. More precisely denoting by $\rho_n$ the discretization step of the observations and assuming that $\rho_n$ goes to zero, we give conditions connecting $\rho_n$ to $N$, the number of Fourier coefficients used in the estimation method. Assuming $N\rho_n \to 0$ as $N, n$ go to infinity, the effect of the discretization disappears and the asymptotic behaviour of the estimator is similar to the one of the estimator based on the observation of the whole path of the process. In this case the consistency of the estimator has been proved in [15]. In this paper we focus on the more realistic case $0 < \lim \inf N\rho_n \leq \lim \sup N\rho_n < +\infty$. In this situation a bias may appear.

Then we apply our general result to particular choices of discretized obervations. In the case of synchronous observations, we show that the estimator of the integrated volatility is consistent as $N$ and $n$ go to infinity without the restriction $N\rho_n \to 0$. Turning to the central limit theorem the situation is drastically different and the asymptotic behavior of $N\rho_n$ is crucial. Indeed if we assume that the sampling is regular and that $N\rho_n$ converges to $a > 0$, we obtain a central limit theorem with rate of convergence $(\rho_n)^{-1/2}$, but the asymptotic variance depends on the parameter $a$. In particular we prove that there are some optimal choices of $N$ with respect to $n$ to minimize the asymptotic variance.

We finally investigate the case of non synchronous data. Although the Fourier coefficients involved in the estimation method are computed separately for each components, the results are rather different
with the situation of synchronous sampling and much more surprising. To illustrate the situation we consider the specific example of an alternate sampling scheme \((n \text{ equidistant data alternatively collected on different components of } X)\). We establish that if \(N \rho_n \) tends to \( a > 0 \) the estimator is not consistent and we give the explicit bias. Then, we find an explicit expression for the variance of the estimator. Moreover, numerical simulations show that the corrected estimator performs well on simulated data.

The paper is organized as follows. The main results are given in Section 2. The convergence of the Fourier estimator as well as its asymptotic normality are established under weak assumptions. Section 3 contains the case of discrete synchronous observations. Finally, in section 4, we consider non synchronous data. The technical proofs are postponed in the Appendix.

2. Main results

In this section, we first recall the definition of the estimator proposed by Malliavin and Man- cino [14]. Then we briefly study the estimator under the framework of an exact observation of the Fourier coefficient of the diffusion. In the final part, we present our main result about the asymptotic of the estimator in the case of a discrete sampling with possibly non-regular and non-synchronous observations.

2.1. Notations and hypotheses

Throughout this paper, we consider a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\) which is the canonical space of a \(d\)-dimensional Brownian motion on the time interval \([0, 2\pi]\). All results could be easily written on any time interval \([0, T]\) but the choice of \([0, 2\pi]\) is convenient in the Fourier context. We denote by \(\mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq 2\pi}\) the usual augmentation of the natural filtration of \(W\) and we denote by \(L^2_a([0, 2\pi])\) the space of real measurable adapted processes \(X = (X(t))_{0 \leq t \leq 2\pi}\) such that \(\mathbb{E} \int_0^{2\pi} |X(t)|^2 dt < +\infty\). We assume that we observe a \(J\)-dimensional continuous time process \(X(t) = (X^1(t), \ldots, X^J(t))\) solution of the equations

\[
\begin{align*}
  dX^j(t) &= b^j(t)dt + \sum_{r=1}^{d} \sigma^{j,r}(t)dW^r(t), \quad \text{for } 1 \leq j \leq J.
\end{align*}
\]

where \(b^j \in L^2_a([0, 2\pi])\) and \(\sigma^{j,r} \in L^2_a([0, 2\pi])\) for \(1 \leq j \leq J\) and \(1 \leq r \leq d\).

Throughout the paper we need some regularity assumptions on the coefficients \(b\) and \(\sigma\). In particular we assume that \(\sigma\) admits a Malliavin derivative. We denote by \(D\) the derivative operator and we refer the reader to Nualart [19] for the basic theory of Malliavin calculus. We make the following hypotheses.
H1. For $1 \leq j \leq J$ and $1 \leq r \leq d$ we assume that $\forall p \geq 1$:

$$
\mathbb{E}(\sup_{t \in [0,2\pi]} |b^j(t)|^p) < +\infty, \quad \mathbb{E}(\sup_{t \in [0,2\pi]} |\sigma^{jr}(t)|^p) < +\infty.
$$

H2. We assume that almost surely the function $t \mapsto \sigma(t)$ is continuous on $[0, 2\pi]$.

H3. We assume that $\forall p \geq 1 \sigma \in \mathbb{D}^{1,p}$ and that for $1 \leq j \leq J$ and $1 \leq r \leq d$

$$
\mathbb{E}(\sup_{s,t \in [0,2\pi]} |D_s \sigma^{jr}(t)|^p) < +\infty.
$$

This hypothesis is satisfied in particular for a diffusion process with smooth coefficients.

In the following, we note $\sigma^j$ the transpose of the vector $\sigma^j$ and more generally $A^\ast$ the transpose of a matrix $A$. We define the volatility process as

$$
\Sigma^{j,j'}(t) = \sum_{r=1}^{d} \sigma^{jr}(t)\sigma^{j'r}(t) = (\sigma^j \ast \sigma^j')(t) \quad (2)
$$

for $1 \leq j, j' \leq J$. Our aim is to study the asymptotic properties of the Fourier transform estimator of $\Sigma$ proposed by Malliavin and Mancino [14]. We denote by $(c_k(\Sigma^{j,j'}))_{k \in \mathbb{Z}}$ the Fourier coefficients of $\Sigma^{j,j'}$, we have:

$$
c_k(\Sigma^{j,j'}) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt}\Sigma^{j,j'}(t)dt. \quad (3)
$$

Moreover for any measurable bounded function $h$, we note $c_k(hdX^j)$ the Fourier coefficient

$$
c_k(hdX^j) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt}h(t)dX^j(t). \quad (4)
$$

With these notations we construct the estimator

$$
\Gamma_N^{j,j'}(h) = \frac{2\pi}{2N+1} \sum_{|l| \leq N} c_{-l}(dX^j)c_l(hdX^{j'}). \quad (5)
$$

We remark that for $h_k(t) = e^{-ikt}$, $\Gamma_N^{j,j'}(h_k)$ is a consistent estimator of $c_k(\Sigma^{j,j'})$ (see [14]).

Let us stress that the quantity (5) might only be considered as an estimator if one exactly observes the Fourier coefficients (4). This seems only feasible from a continuous observation of the sample path of the diffusion, and thus, seems impossible from a practical point of view. Nevertheless, we will briefly study (as a start) this situation, before turning to a more realistic one.

2.2. Preliminary case: continuous observation of the path

The consistency of the estimator has been proved in [14]–[15]. Here, we study a central limit theorem, with stable convergence in law. We recall that the stable convergence in law of a sequence $(U_n)$ is equivalent to the convergence in law of the couple $(Y, U_n)$, for all $\mathcal{F}$–measurable random variable $Y$. We refer to Jacod [11] for a complete definition of the stable convergence in law.
Proposition 1. Under $H_1$, $H_2$ and $H_3$, for all $(j, j') \in \{1, \ldots, J\}^2$, the sequence
\[
\sqrt{N}(\Gamma_N^{j,j'}(h) - \frac{1}{2\pi} \int_0^{2\pi} h(t) \Sigma^{j,j'}(t) dt)
\]
converges stably in law, as $N \to \infty$, to a variable with the representation
\[
\frac{1}{2\sqrt{\pi}} \int_0^{2\pi} |h(s)| \sqrt{(\sigma^j\sigma^j)(s)(\sigma^{j'}\sigma^{j'})(s) + (\sigma^j\sigma^{j'})^2(s)} d\tilde{W}(s),
\]
where $\tilde{W}$ is a Brownian motion independent of $W$.

This result shows that the rate of estimation is the square root of the number of Fourier coefficients used in the method, and the asymptotic random variance is explicit.

Let us stress that an extension of this result to an unbounded function $h$ or to $\delta$-functions is not trivial and consequently, we cannot derive some limit theorems for the spot volatility from our results.

The proof of Proposition 1 is postponed to the Appendix. Nevertheless, let us remark that the expression of the variance is connected to the $L^2$-norm of the Dirichlet kernel. In fact applying Ito’s formula to the product of stochastic integrals $c_{-t}(dX^j)c_{t}(hdX^{j'})$ and using (5) we have:
\[
\Gamma_N^{j,j'}(h) = \frac{1}{2\pi} \int_0^{2\pi} h(t) \Sigma^{j,j'}(t) dt + R_N^{j,j'}(2\pi),
\]
\[
R_N^{j,j'}(t) = \frac{1}{2\pi} \left( \int_0^t \left( \int_0^s d_N(s-u)dX^j(u)h(s)dX^{j'}(s) + \int_s^t \left( \int_0^u d_N(s-u)h(u)dX^{j'}(u)dX^j(s) \right) \right) \right),
\]
where $d_N$ denotes the normalized Dirichlet kernel
\[
d_N(u) = \frac{1}{2N+1} \sum_{k=-N}^N e^{iku} = \frac{1}{2N+1} \frac{\sin((2N+1)u/2)}{\sin(u/2)}.\]
Hence, Proposition 1 is clearly equivalent to the convergence of $\sqrt{N}R_N^{j,j'}(2\pi)$, where the Dirichlet kernel plays a crucial role.

2.3. Discrete observation of the process

In this section, we assume that we observe a discrete sampling of the Itô process $X$. We might consider the case where the instants of observation are different for each components of the process. Our goal is to estimate the integrated co-volatility between the components $X^j$ and $X^{j'}$ with $j, j' \in \{1, \ldots, J\}$. If $j = j'$ we estimate the integrated volatility of $X^j$. To lighten the notations we assume that $j = 1$ and $j' = 2$ but it does not exclude the estimation of the integrated volatility assuming $X^1 = X^2$. We respectively denote by $(t^1_k)_{k=0}^{M^1}$ and $(t^2_k)_{k=0}^{M^2}$ the sampling times for $X^1$ and
The estimator is based on the discrete Fourier coefficients:

\[ c_k^{n,1}(dX^1) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\phi(t)} dX^1(t), \quad c_k^{n,2}(h_n dX^2) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\phi(t)} h(\phi^2_n(t)) dX^2(t) \]  

(9)

where \( h_n(t) = h(\phi^2_n(t)) \). We shall focus on the covolatility estimator

\[ \Gamma_{N,n}^{1,2}(h) = \frac{2\pi}{2N + 1} \sum_{|l| \leq N} c_{-l}^{n,1}(dX^1) c_{l}^{n,2}(h_n dX^2). \]  

(10)

However, it is clear that if one takes \( X^2 = X^1 \) and \( \phi^2_n(t) = \phi^1_n(t) \), then the covolatility estimator reduces to the estimator of the volatility of the single component \( X^1 \).

As suggested by the previous section, the Dirichlet kernel (8) will play a crucial role. To maintain shorter notation, we define for \( s, u \in [0, 2\pi] \):

\[ d_{N,n}^{ij}(s, u) = d_N(\phi^j_n(s) - \phi^j_n(u)), \quad \text{with } i, j \in \{1, 2\}. \]  

(11)

Naturally, we have the simple relation \( d_{N,n}^{12}(s, u) = d_{N,n}^{21}(u, s) \).

We make the following assumptions.

**A1** We note \( \rho_n = \sup_{j=1,2} \sup_{k=0,\ldots,M_n^{\ell}} |t_{k+1}^j - t_k^j| \), and we assume that \( \rho_n \xrightarrow{n \to \infty} 0 \).

**A2** We assume that \( N, n \to \infty \) such that the sequence of functions \( t \mapsto d_{N,n}^{12}(t, t) \) weakly converges to some integrable function \( \gamma(t) \) : for all \( t \in [0, 2\pi] \),

\[ \int_0^t d_{N,n}^{12}(s, s) ds \xrightarrow{N, n \to \infty} \int_0^t \gamma(s) ds. \]

Remark that A2 clearly implies that if \( f \) is a continuous function then \( \int_0^{2\pi} d_{N,n}^{12}(s, s) f(s) ds \) converges to \( \int_0^{2\pi} \gamma(s) f(s) ds \).

**Proposition 2.** Let \( h \) be a continuous bounded function. Assume \( H1, H2, A1 \) and \( A2 \), then we have:

\[ \Gamma_{N,n}^{1,2}(h) \xrightarrow{N, n \to \infty} \frac{1}{2\pi} \int_0^{2\pi} \gamma(t) h(t) \Sigma^{1,2}(t) dt. \]

**Proof** Applying Ito’s formula to the product \( c_k^{n,1}(dX^1) c_k^{n,2}(h_n dX^2) \) we obtain a decomposition analogous to (6)–(7) in the case of discrete data:

\[ \Gamma_{N,n}^{1,2}(h) = \frac{1}{2\pi} \int_0^{2\pi} d_{N,n}^{12}(t, t) h_n(t) \Sigma^{1,2}(t) dt + R_{N,n}^{1,2}(2\pi), \]  

(12)
where $R_{N,n}^{1,2}$ is the process defined by,

$$R_{N,n}^{1,2}(t) = \frac{1}{2\pi} \left( \int_{0}^{t} \int_{0}^{s} d_{N,n}^{1,2}(u,s) dX(u,s) h_{n}(s) dX^{2}(s) + \int_{0}^{t} \int_{0}^{s} d_{N,n}^{2,1}(u,s) h_{n}(u) dX^{2}(u) dX^{1}(s) \right)$$

(13)

and $d_{N,n}^{i,j}$ is given by (11).

Using the continuity of the functions $h$ and $\Sigma^{1,2}$, the assumptions A1–A2 easily imply that the first term in the right hand side of (12) converges almost surely to $\frac{1}{2\pi} \int_{0}^{2\pi} \gamma(t) h(t) \Sigma^{1,2}(t) dt$.

It remains to check that $R_{N,n}^{1,2}(2\pi)$ converges to zero in $L^{2}$ norm. Since $X$ is solution of (1) we have the decomposition

$$R_{N,n}^{1,2}(t) = M_{N,n}(t) + \tilde{M}_{N,n}(t) + I_{N,n}^{1}(t) + I_{N,n}^{2}(t) + I_{N,n}^{3}(t) + \tilde{I}_{N,n}^{1}(t) + \tilde{I}_{N,n}^{2}(t) + \tilde{I}_{N,n}^{3}(t),$$

(14)

where the first two terms are martingale processes,

$$M_{N,n}(t) = \frac{1}{2\pi} \int_{0}^{t} \left( \int_{0}^{s} d_{N,n}^{1,2}(s,u) \sigma^{1*}(u) dW(u) \right) h_{n}(s) \sigma^{2*}(s) dW(s),$$

(15)

$$\tilde{M}_{N,n}(t) = \frac{1}{2\pi} \int_{0}^{t} \left( \int_{0}^{s} d_{N,n}^{2,1}(s,u) h_{n}(u) \sigma^{2*}(u) dW(u) \right) \sigma^{1*}(s) dW(s),$$

(16)

and the other ones are the contribution of the drift part. We clearly have,

$$I_{N,n}^{1}(t) = \frac{1}{2\pi} \int_{0}^{t} \left( \int_{0}^{s} d_{N,n}^{1,2}(s,u) b^{1}(u) dW(u) \right) h_{n}(s) \sigma^{2*}(s) dW(s),$$

(17)

$$I_{N,n}^{2}(t) = \frac{1}{2\pi} \int_{0}^{t} \left( \int_{0}^{s} d_{N,n}^{1,2}(s,u) \sigma^{1*}(u) dW(u) \right) h_{n}(s) b^{2}(s) ds,$$

(18)

$$I_{N,n}^{3}(t) = \frac{1}{2\pi} \int_{0}^{t} \left( \int_{0}^{s} d_{N,n}^{1,2}(s,u) b^{1}(u) dW(u) \right) h_{n}(s) b^{2}(s) ds,$$

(19)

and $\tilde{I}_{N,n}^{1}, \tilde{I}_{N,n}^{2}, \tilde{I}_{N,n}^{3}$ are defined in a symmetric way. We finish the proof by showing that these terms converge to zero. Using assumption H1 with the Burkholder–Davis–Gundy inequality we get

$$E \left[ (M_{N,n}(2\pi))^{2} \right] \leq C \int_{0}^{2\pi} \int_{0}^{s} d_{N,n}^{1,2}(s,u)^{2} duds,$$

(20)

for some constant $C$.

Assume that $u \neq s$ are fixed, then by A1, for $n$ large enough, we have $|\varphi_{n}^{1}(u) - \varphi_{n}^{2}(s)| > |u - s|/2 > 0$. Using that the Dirichlet kernel $d_{N}$ converges uniformly to zero on compact subsets of $(0,2\pi)$ we deduce that $d_{N,n}^{1,2}(s,u) \xrightarrow{n\to\infty} 0$. Since $d_{N,n}^{1,2}(s,u)$ is bounded by the constant 1, the dominated convergence theorem implies that the right hand side of (20) converges to zero.
We treat $\tilde{M}_{N,n}(2\pi)$ analogously. For the contribution of the drift, we focus on $I_{N,n}^1$ since the proof for the other terms is similar or easier. From $H1$ again, we have

$$E I_{N,n}^1(t)^2 = \frac{1}{(2\pi)^2} \int_0^t E \left( \left( \int_0^s d_{N,n}^{1,2}(s,u) b^1(u) du \right)^2 h_n(s)^2 (\sigma^2 \sigma^2)(s) \right) ds,$$

$$\leq C \int_0^{2\pi} \left( \int_0^s |d_{N,n}^{1,2}(s,u)|^2 du \right)^2 ds. \quad (21)$$

The latter integral is clearly bounded by the right hand side of (20) and thus converges to zero. This completes the proof of the proposition.

\[ \Diamond \]

Remark 1. Let us stress that the conditions A1–A2 for convergence of the estimator are rather weak:

1) If the samplings are synchronous then $\varphi_n(t) = \varphi_n^2(t)$ and $d_{N,n}^{1,2}(t,t) = 1$. Thus, conditions A1–A2 reduce to $\rho_n \to 0$ and $N \to \infty$. Moreover, here $\gamma \equiv 1$, and $\Gamma_{N,n}^{1,2}(1)$ is a consistent estimator for the co-volatility.

2) In a general situation the condition A2 relates the choice of the frequency $N$ with the sampling scheme. It is always possible, by choosing $1/N$ converging slowly to zero, to get that $d_{N,n}^{1,2}(t,t)$ converges pointwise to 1. A sufficient condition is $N \rho_n \xrightarrow{N,n \to \infty} 0$. In this case, A2 holds with $\gamma \equiv 1$, and the estimator $\Gamma_{N,n}^{1,2}(1)$ is consistent for the co-volatility.

3) If $N$ is chosen larger, a bias appears when $d_{N,n}^{1,2}(t,t)$ converges to a function not everywhere equal to one. We shall see that the weak convergence of $d_{N,n}^{1,2}(t,t)$ is the natural assumption in this circumstance.

However, since $\gamma$ can be computed from the sampling scheme, it is possible to construct a consistent estimator for the co-volatility by choosing $h = 1/\gamma$. An example of such a situation is given in Section 4.

Before stating a central limit theorem related to Proposition 2, we add the following assumptions.

A3 We assume that $N, n$ are such that

$$0 < \lim \inf N \rho_n \leq \lim \sup N \rho_n < \infty.$$  

The assumption A3 excludes the simpler choice $N \rho_n \to 0$. Indeed, in such situation the estimator uses a small number of Fourier coefficients compared to the number of available data. But Proposition 1 asserts that, in the case of a continuous observation, the rate is $\sqrt{N}$. Hence, if $N \rho_n \to 0$ the rate of the estimator is limited by the number of Fourier coefficients rather than by the number of observations.

A4 We assume that $N, n$ are such that there exist three integrable functions $\gamma^{1,2}, \gamma^{1,2}, \gamma^c$ defined
on \([0, 2\pi]\) such that, the following convergences hold for all \(t \in [0, 2\pi]\),

\[
\rho_n^{-1} \int_0^t \int_0^s d_{N,n}^{1,2}(s,u)^2 duds \xrightarrow{N,n \to \infty} \int_0^t \gamma^{1,2}(s) ds,
\]

\[
\rho_n^{-1} \int_0^t \int_0^s d_{N,n}^{2,1}(s,u)^2 duds \xrightarrow{N,n \to \infty} \int_0^t \gamma^{1,2}(s) ds,
\]

\[
\rho_n^{-1} \int_0^t \int_0^s d_{N,n}^{1,2}(s,u) d_{N,n}^{2,1}(s,u) duds \xrightarrow{N,n \to \infty} \int_0^t \gamma^c(s) ds.
\]

This assumption is analogous to A2 and is necessary to obtain a central limit theorem. It relates the observations times to \(N\) and need to be checked for each sampling scheme (see sections 3 and 4).

**Theorem 1.** Let \(h\) be a continuous bounded function. Assume H1–H3, A1, A3–A4, then the sequence of random variables

\[
\rho_n^{-1/2} \left( \Gamma_{1,2}^{1,2}(h) - \frac{1}{2\pi} \int_0^{2\pi} d_{N,n}^{1,2}(t,t) h_n(t) \Sigma^{1,2}(t) dt \right)
\]

converges stably in law to a variable with the representation,

\[
\frac{1}{2\pi} \int_0^{2\pi} |h(t)| \sqrt{(\gamma^{1,2}(t) + \tilde{\gamma}^{1,2}(t)) \Sigma^{1,1}(t) \Sigma^{2,2}(t) + 2\gamma^c(t)(\Sigma^{1,2}(t))^2} d\tilde{W}_t.
\]

**Remark 2.** The rate of convergence of the estimator is given as a function of \(\rho_n^{-1}\), which, in most situations, is of the same magnitude as the number of data.

**Remark 3.** Actually, if we choose \(N\) with \(N\rho_n \to 0\), we can prove that the rate is \(N^{-1/2}\) and the limit variable is the same as in Proposition 1.

The proof of Theorem 1 is given in Appendix.

3. Application to synchronous data

In this section we apply the results of the previous section to the case of synchronous data.

3.1. Explicit expressions in the case of a regular sampling

We assume in this section that we observe the process \(X(t)\) solution of (1) at time \(t_k = \frac{2\pi k}{n}\) for \(k = 0, \ldots, n\). We denote by \(\varphi_n\) the function:

\[
\varphi_n(t) = \frac{2\pi k}{n} \quad \text{if} \quad \frac{2\pi k}{n} \leq t < \frac{2\pi (k + 1)}{n}.
\]

Our estimators are based on the discrete Fourier coefficients

\[
c_k^n(h_n dX^j) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\varphi_n(t)} h_n(t) dX^j(t), \quad j = 1, \ldots, J.
\]

where \(h_n(t) = h(\varphi_n(t))\).
With these notations we define the discrete time estimators of the co-variation between the components \(X^j\) and \(X^{j'}\),

\[
\Gamma_{N,n}^{j,j'}(h) = \frac{2\pi}{2N + 1} \sum_{|l| \leq N} c_l^{j}(dX^j)c_l^{j'}(h_n dX^{j'}). \tag{24}
\]

**Theorem 2.** Let \(h\) be a continuous bounded function with bounded derivative. We assume that \(N\) and \(n\) tend to infinity and that \(\lim N/n = a > 0\) then under \(H1, H2\) and \(H3\) for all \(j, j' \in \{1, \ldots, J\}\), we have:

\[
\sqrt{n} \left( \Gamma_{N,n}^{j,j'}(h) - \frac{1}{2\pi} \int_0^{2\pi} h(t) \Sigma^{j,j'}(t) dt \right) \text{ converges stably in law to a variable with the representation}
\]

\[
\frac{1}{\sqrt{2\pi}} \left( 1 + 2\eta(2a) \right)^{1/2} \int_0^{2\pi} |h(s)| \sqrt{\Sigma^{j,j'}(s)\Sigma^{j',j}(s) + \Sigma^{j,j'}(s)^2} d\tilde{W}(s),
\]

where \(\eta(a) = \frac{1}{2\pi}r(a)(1 - r(a))\) and \(r(a) = a - [a]\) is the fractional part of \(a\).

**Proof** To lighten the notations we take \(j = 1, j' = 2\) and we intend to use Theorem 1. Since the sampling is regular we have \(\varphi_1^n(t) = \varphi_2^n(t) = \varphi_n(t)\) and \(d_{N,n}^{1,2}(t, t) = d_N(0) = 1\). Moreover, the function \(h\) is smooth and we easily get

\[
\frac{1}{2\pi} \int_0^{2\pi} d_{N,n}^{1,2}(t, t)h_n^2(t)\Sigma_1^2(t) dt = \frac{1}{2\pi} \int_0^{2\pi} h(t)\Sigma_1^2(t) dt + o(n^{-1/2}).
\]

Hence the result follows from a direct application of Theorem 1. Assumptions A1 and A3 are checked immediately with \(\rho_n = 2\pi/n\), and condition A4 follows from Lemma 1 below whose proof is given in the Appendix.

**Lemma 1.** Let \(\varphi_n\) defined by (22) and \(d_N\) the normalized Dirichlet kernel then we have, assuming that \(N/n \to a > 0\):

\[
\forall 0 < t < 2\pi, \lim_{N,n \to \infty} n \int_0^t \int_0^s d_N(\varphi_n(s) - \varphi_n(u))^2 du ds = t\pi(1 + 2\eta(2a)) \tag{25}
\]

with

\[
\eta(a) = \sum_{k=1}^{\infty} \frac{\sin^2(a\pi k)}{(a\pi k)^2} = \frac{1}{2a^2}r(a)(1 - r(a)), \quad r(a) = a - [a], \tag{26}
\]

where \([x]\) denotes the integer part of \(x\).

**Remark 4.** We remark that \(\eta(a) = 0\) for \(a \in \mathbb{N}^*\), so, we deduce that the optimal asymptotic variance in Theorem 2 is obtained for \(2a \in \mathbb{N}^*\) and is equal to

\[
\frac{1}{2\pi} \int_0^{2\pi} h(s)^2 \left( \Sigma^{j,j'}(s)\Sigma^{j',j}(s) + \Sigma^{j,j'}(s)^2 \right) ds.
\]

In particular given \(n\) the number of observations, the choice of \(N = n/2\) Fourier coefficients to estimate the integrated volatility is optimal and in this case, the variance is the same one as for the quadratic covariation estimator (see Genon-Catalot and Jacod [5]). Remark that the choice \(N = n/2\) was used in earlier empirical work [17] since it corresponds to the natural choice of the Nyquist frequency of the signal.
Remark 5. The case \(a = 0\) is excluded in Theorem 2. In fact, \(\eta(2a) \sim 1/(4a)\) as \(a \to 0\) and consequently the asymptotic variance in Theorem 2 goes to infinity. This is coherent with the rate and the variance of the continuous case (see Remark 3).

It might seem surprising that the variance in Theorem 2 becomes optimal, again, when \(a \to \infty\). Indeed, in this situation the estimator becomes close to the quadratic covariation of the process \((X^j, X^{j'})\). In fact if we note \(\Delta X^j_k = X^j(t_{k+1}) - X^j(t_k)\), we have

\[
c_n^{-1}(dX^j) c_n^{-1}(dX^{j'}) = \frac{1}{4\pi^2} \sum_{k=0}^{n-1} e^{it \frac{2k\pi}{n}} \Delta X^j_k \sum_{k=0}^{n-1} e^{-it \frac{2k\pi}{n}} \Delta X^{j'}_k,
\]

\[
= \frac{1}{4\pi^2} \sum_{k=0}^{n-1} \Delta X^j_k \Delta X^{j'}_k + \frac{1}{4\pi^2} \sum_{k \neq k'=0}^{n-1} e^{it \frac{2(k-k')\pi}{n}} \Delta X^j_k \Delta X^{j'}_{k'}.
\]

This gives

\[
\Gamma^j_{N,n}(1) = \frac{1}{2\pi} \sum_{k=0}^{n-1} \Delta X^j_k \Delta X^{j'}_k + \frac{1}{2\pi} \sum_{k \neq k'=0}^{n-1} d_N(\frac{2(k-k')\pi}{n}) \Delta X^j_k \Delta X^{j'}_{k'},
\]

Assuming that \(n\) is fixed and letting \(N\) go to infinity, we have \(\lim_N d_N(\frac{2(k-k')\pi}{n}) = 0\), for \(k \neq k'\) and consequently

\[
\lim_{N \to \infty} \Gamma^j_{N,n}(1) = \frac{1}{2\pi} \sum_{k=0}^{n-1} \Delta X^j_k \Delta X^{j'}_k \quad (n \text{ fixed}) \tag{27}
\]

which is the expected result.

As a conclusion, the value of \(a\) enables us to calibrate the estimator between the exact Fourier based estimator \((N/n \to 0)\) and the usual quadratic variation one \((N/n \to \infty)\). It is noteworthy that intermediate values of \(a = \lim N/n\) are optimal for the variance.

3.2. Case of irregular sampling

In the case of a synchronous, but irregular sampling, of the process at the instants \(t_i\) for \(i = 0, \ldots, M_n\), it is clear that checking the condition A4 is cumbersome. However one can easily see that, under A3, the condition A4 reduces to the convergence for all \(t\),

\[
\rho_n^{-1} \sum_{i | t_i \leq t} \frac{(t_{i+1} - t_i)^2}{2} + \rho_n^{-1} \sum_{i | t_i \leq t} \sum_{j=0}^{i-1} d_N(t_i - t_j)^2(t_{i+1} - t_i)(t_{j+1} - t_j) \xrightarrow{N,n \to \infty} \int_0^t \gamma^{1,2}(s)ds. \tag{28}
\]

Hence the condition A4 is somehow related to the quadratic variation in time condition of [18] which is

\[
\tilde{\rho}_n^{-1} \sum_{i | t_i \leq t} \frac{(t_{i+1} - t_i)^2}{2} \xrightarrow{n \to \infty} \frac{h(t)}{2} = \frac{1}{2} \int_0^t h'(s)ds, \tag{29}
\]

for some differentiable function \(h\) and some rate \(\tilde{\rho}_n\).
For instance, if the sampling is recursively defined as \( t_{i+1} = t_i + \frac{h'(t_i)}{n} \) for \( h' > 0 \) a given continuous function, then it is clear that the quadratic variation in time condition (29) holds with \( \tilde{\rho}_n = 1/n \). In this situation, and with \( N/n \to a > 0 \), one could see that (28) holds true with the more convenient re-normalization term \( \tilde{\rho}_n = 1/n \) and \( \gamma^{1,2}(s) = \frac{1}{2} h'(s) + h'(s) \eta\left( \frac{ah'(s)}{\pi} \right) \). Remark that the change of normalization is just for notational convenience since \( \rho_n \sim n^{\inf h'} \).

3.3. Numerical simulations

We study the behaviour of the estimator on simulated data with a Monte Carlo method.

We consider the framework of Section 3.1 and assume, for simplicity, that \( X_t = B_t \). The data are collected at the instants \( i2\pi/n \) with \( i = 0, \ldots, n-1 \) and the estimator is \( \Gamma_{N,n}^{1,1} = \Gamma_{N,n}^{1,1}(h) \) given by (24) with \( h \equiv 1 \). Hence, we are estimating the integrated volatility \( \frac{1}{2\pi} \int_0^{2\pi} \Sigma_s^{1,1} ds \) which is equal, here, to the constant 1. The Figure 1 shows the mean value and the standard deviation of the estimator in the case \( n = 100 \) and for values of \( N \) ranging from \( N = 20 \) to \( N = 1000 \). We made 10000 replications. It is clear that the estimator has no bias whatever is the choice of \( N \). According to Theorem 2 the optimal choices of \( N \), for minimizing the variance, are such that \( 2a \approx 2N/n \in \mathbb{N}_* \). We see that if \( N < n/2 = 50 \), the standard deviation increases while \( N \) decreases. This is natural since the variance in Theorem 2 explodes as \( a \to 0 \), actually we know that when \( N \) is too small (\( a \approx 0 \)) the right rate of estimation is \( 1/\sqrt{N} \) (see remark 5). The choice \( N = 50 \) corresponds to \( a = 1/2 \) and is optimal, moreover, on the Figure 1 the standard deviation seems almost constant for \( N > n/2 = 50 \). In graph 2 we plot the variance of the estimator multiplied by \( n \) for \( N = 50 \) to \( N = 1000 \). We, now, clearly see that \( 2N/n \in \mathbb{N}_* \) give minimal variances and the deviations are a bit higher otherwise. However, as \( N/n \) becomes very large, the variance of the estimator tends to the optimality, which is natural since the estimator becomes similar to the quadratic variation estimator (recall (27)).

4. A tractable example of non synchronous data: alternate sampling

We assume that \( (X^1(t)) \) is observed at time \( t_k^1 = 2k\pi/n \) for \( k = 0, \ldots, n \), and thus \( \varphi_n^1(t) = 2k\pi/n \) if \( 2k\pi/n \leq t < 2(k+1)\pi/n \). The process \( (X^2(t)) \) is observed at time \( t_k^2 = 2\pi k/n + \pi/n \), for \( k = 0, \ldots, n-1 \), yielding to \( \varphi_n^2(t) = 2k\pi/n + \pi/n \), for \( 2k\pi/n + \pi/n \leq t < 2(k+1)\pi/n + \pi/n \). We also assume that we observe \( X^2(0) \) and \( X^2(2\pi) \). This situation of alternate sampling is studied in [7] too. Although being a particular case of non synchronous observation, its main advantage is that all computations are explicitly tractable here, and a comparison with the results in [7] will be possible.
4.1. Behaviour of the estimator

Our aim is to estimate the covolatility $\Sigma_{1,2}^1 = (\sigma_1^1, \sigma_2^2)(t)$ and we consider the Fourier estimator $\Gamma_{N,n}^{1,2}(h)$ defined by the equations (9)–(10). We first establish that the estimator $\Gamma_{N,n}^{1,2}(h)$ is not consistent but we have an explicit bias.

**Theorem 3.** Let $h$ be a continuous function then, under $H1$–$H2$, $\Gamma_{N,n}^{1,2}(h)$ converges in probability to

$$\frac{1}{2\pi} \int_0^{2\pi} h(t) \Sigma_{1,2}(t) dt \frac{\sin(a\pi)}{a\pi}$$

as $N$ and $n$ go to infinity and $N/n$ goes to $a \in \mathbb{R}_+$. If $a = 0$, which means that the number of observations $n$ goes faster to infinity than the number of Fourier coefficients $N$, the effect of non synchronous data disappear and we have a consistent estimator of the integrated volatility $\frac{1}{2\pi} \int_0^{2\pi} h(t) \Sigma_{1,2}(t) dt$. If $a \in \mathbb{N}^*$, $\Gamma_{N,n}^{1,2}(h)$ goes to zero and the estimation method seems useless. Finally if $a \not\in \mathbb{N}$, a consistent estimator is given by $\frac{a\pi}{\sin(a\pi)} \Gamma_{N,n}^{1,2}(h)$.

**Proof** Using the Proposition 2 we just determine the limit of $\int_0^t d_{N,n}^{1,2}(s,s) ds$ for any $t$. One can easily see that

$$\int_{2k\pi/n}^{2(k+1)\pi/n} d_{N,n}^{1,2}(s,s) ds = d_N(-\frac{\pi}{n})\pi/n + d_N(\frac{\pi}{n})\pi/n,$$

and consequently

$$\int_0^t d_{N,n}^{1,2}(s,s) ds = d_N(\frac{\pi}{n})t + o(1/n).$$

We conclude the proof observing that $d_N(\frac{\pi}{n})$ converges to $\frac{\sin(a\pi)}{a\pi}$.

**Remark 6.** We could treat, in the same way, more general non synchronous sampling. Indeed if $X^1$ is sampled at times $t^1_k = 2k\pi/n$ and $X^2$ at times $t^2_k = (2k + s)\pi/n$, where $s$ in any fixed shift in $(0,1)$, we can show that the bias factor becomes $s\frac{\sin(2\pi(1-s)a)}{2\pi(1-s)a} + (1-s)\frac{\sin(2\pi sa)}{2\pi sa}$. However the associated central limit theorem is not explicit except if $s = 1/2$.

We study now the rate of convergence of the estimator $\Gamma_{N,n}^{1,2}(h)$ and give some explicit limit.

**Theorem 4.** We assume that $H1$, $H2$ and $H3$ hold true, that $h$ is a function with a bounded derivative, and that $N$ and $n$ go to infinity such that $N/n = a + o(1/\sqrt{n})$ as $n \to \infty$, with $a > 0$. Then

$$\sqrt{n}(\Gamma_{N,n}^{1,2}(h) - \frac{1}{2\pi} \int_0^{2\pi} h(t) \Sigma_{1,2}(t) dt \frac{\sin(a\pi)}{a\pi})$$

cconverges stably in law to

$$\frac{1}{\sqrt{2\pi}} \int_0^{2\pi} |h(s)| \gamma_1(a)\Sigma_{1,2}(1)\Sigma_{2,1}(s) + \gamma_2(a)\Sigma_{1,2}(s)\Sigma_{2,1}(s) + \gamma_3(a)\Sigma_{1,2}(s)\Sigma_{2,1}(s) dW(s),$$

13
where $\bar{W}$ is a brownian motion independent of $W$ and where $\gamma_1$ and $\gamma_2$ are defined by
\[
\gamma_1(a) = 2\pi(\eta(a) - \eta(2a)),
\]
\[
\gamma_2(a) = \pi(\eta(a) - \eta(2a)) + \frac{\sin 2\pi a}{8a^2} (1_{(0,1/2)}(r(a)) - 1_{(1/2,1)}(r(a))).
\]
Remark that for $a \in \mathbb{N}$ we have $\gamma_1(a) = \gamma_2(a) = 0$ and the estimator $\Gamma_{N,n}^{1,2}(h)$ converges (to zero) faster than the rate $\sqrt{n}$.

**Proof** We will use Theorem 1. First, observe that
\[
\frac{1}{2\pi} \int_0^{2\pi} d_{N,n}^{1,2}(t,t) h_n^2(t) \Sigma_1(t) dt = d_N(\frac{\pi}{n}) \frac{1}{2\pi} \int_0^{2\pi} h_n^2(t) \Sigma_1(t) dt,
\]
consequently,
\[
\frac{1}{2\pi} \int_0^{2\pi} d_{N,n}^{1,2}(t,t) h_n^2(t) \Sigma_1(t) dt - \frac{1}{2\pi} \int_0^{2\pi} h(t) \Sigma_1(t) dt \frac{\sin(\pi a)}{\pi a} = o(\frac{1}{\sqrt{n}})
\]
as soon as
\[
d_N(\frac{\pi}{n}) \frac{\sin(\pi a)}{\pi a} = o(\frac{1}{\sqrt{n}}),
\]
which is equivalent to the condition given in the statement of the theorem: $N/n - a = o(\frac{1}{\sqrt{n}})$.

Now, using (32), the theorem is a straightforward application of Theorem 1: the condition A3 is satisfied since $n\rho_n \rightarrow 2\pi a > 0$ and the condition A4 is checked in Lemma 2.

\[\Box\]

**Lemma 2.** Assume that $N$ and $n$ go to infinity with $N/n \rightarrow a > 0$ then we have $\forall t \in [0,2\pi[ :$
\begin{enumerate}
  \item $\lim_{N,n} \int_0^t \int_0^s d_{N,n}^{1,2}(s,u) \Sigma_1 du = \gamma_1(a)t,$
  \item $\lim_{N,n} \int_0^t \int_0^s d_{N,n}^{2,1}(s,u) \Sigma_1 du = \gamma_1(a)t,$
  \item $\lim_{N,n} \int_0^t \int_0^s d_{N,n}^{2,2}(s,u) \Sigma_1 du = \gamma_2(a)t,$
\end{enumerate}
where $\gamma_1$ and $\gamma_2$ are defined in Theorem 4.

The proof of Lemma 2 is given in the Appendix.

### 4.2. Numerical results and comparison with Hayashi and Yoshida estimator

To have simpler expressions, we restrict ourself to the case of a constant co-volatility. Hence we set $X_t^1 = B_t$ and $X_t^2 = B_t$ for $t \in [0,2\pi]$, yielding to the constant integrated covolatility $\frac{1}{2\pi} \int_0^{2\pi} \Sigma_1 dt = 1$.

The process $X^1$ (respectively $X^2$) is observed at the instants $2i\pi/n$ (resp. $(2i+1)\pi/n$). The estimator $\Gamma_{N,n}^{1,2} := \Gamma_{N,n}^{1,2}(h)$ given in (9)–(10) with $h \equiv 1$ is biased by Theorem 3, thus we define a corrected version:
\[
\Gamma_{N,n}^{\text{unbiased}} = \left( \frac{\sin(\pi N/n)}{\pi N/n} \right)^{-1} \Gamma_{N,n}^{1,2}
\]
when \( \sin(\pi N/n) \neq 0 \). If \( N/n \to a > 0 \) and \( a \notin \mathbb{N} \), then, by Theorem 3, \( \Gamma_{N,n}^{\text{unbiased}} \) is a consistent estimator of the co-volatility. Moreover, Theorem 4 with simple computations implies that, \( \sqrt{n}(\Gamma_{N,n}^{\text{unbiased}} - 1) \xrightarrow{n,N \to \infty} \mathcal{N}(0,v(a)) \) where

\[
v(a) = \frac{3\pi^2}{4\sin(\pi a)^2} (r(a)1_{(0,1/2]}(r(a)) + (1 - r(a))1_{(1/2,1]}(r(a))) + \frac{\pi \sin(2\pi a)}{8 \sin(\pi a)^2} (1_{(0,1/2]}(r(a)) - 1_{(1/2,1]}(r(a)))
\]

The variance of the estimator, \( v(a) \), is a 1-periodic function of \( a \) and it is easy to check that \( v(a) \sim_{a \to 0} 1/a \), which reflects that \( \sqrt{n} \) is not the right rate of convergence for \( a = 0 \). As \( a \to 1^- \), we readily have \( v(a) \sim 1/(1 - a) \), which shows that the divergence of the factor \( \frac{\pi a}{\sin(\pi a)} \) is partially compensated by the cancellation of \( \gamma_1(a) \) and \( \gamma_2(a) \) for \( a = 1 \) (recall (30)–(31)). In Figure 3, we plot the graph of the function \( v \) for \( a \in (0,1) \). Contrarily to the synchronous regular case, the choice \( a = 1/2 \) does not give the minimum variance, however \( v(1/2) = 3\pi^2/8 \approx 3.70 \) remains close to the minimum of the function \( v \). Numerically, the variance is minimized for \( a \approx 0.42 \) and \( a \approx 0.58 \) where \( v(a) \approx 3.52 \). This is slightly larger than the theoretical variance, \( 7/2 \), of the ‘non synchronous covariance estimator’ introduced by Hayashi and Yoshida [7] [8]. We conclude that, in this case, the Fourier covariance estimator is never efficient.

In Figure 4 we show the empirical mean and standard deviation of the estimator for \( n = 100 \) and with \( N \) ranging from 10 to 90 (out of a Monte Carlo study with 10000 replications). It is clear that the estimator is unbiased and the standard deviation remains low on all this range of choice for \( N \). In Figure 5, we plot the variance multiplied by \( n \) for \( N = 10 \) to \( N = 90 \). The graph shows a good concordance with the theoretical values shown in Figure 3. On the whole, it appears that the estimator performs well for estimating the integrated co-volatility and is not very sensitive to the choice of \( N \in \{30,\ldots,70\} \).

Another conclusion of this case study is that using the non-corrected version of the estimator (10) would give poor results. Indeed, keeping the bias low would yield to a choice of the ratio \( N/n \) close to zero, which in turn deteriorates the variance of estimation.

5. Appendix

5.1. Proof of Theorem 1

The proof of Theorem 1 is based on the two following lemmas.
Lemma 3. Assume $A1$ and $A3$, then,

$$\forall p > 1, \exists C_p, \limsup_{n,N \to \infty} \sup_{s \in [0,2\pi]} \int_0^{2\pi} |d_{N,n}^{1,2}(s,u)|^p + |d_{N,n}^{2,1}(s,u)|^p \, du \leq C_p.$$  

Proof By definition (11) it is sufficient to evaluate $\sup_{c \in [0,2\pi]} \int_0^{2\pi} |d_N(\varphi_n^i(u) - c)|^p \, du$. But from the periodicity of the Dirichlet kernel, one can see that it is sufficient to get the bound

$$\limsup_{n,N \to \infty} \rho_n^{-1} \sup_{c \in \mathbb{C}} \int_{(c,\pi) \cap 0}^{(\pi+c) \cap 2\pi} |d_N(\varphi_n^i(u) - c)|^p \, du < \infty.$$  

Temporarily denote $a = \limsup N \rho_n$ which is finite by $A3$. We split the integral above as

$$\int_{(c,\pi) \cap 0}^{(\pi+c) \cap 2\pi} 1_{|u-c| > \frac{2aN}{N}} |d_N(\varphi_n^i(u) - c)|^p \, du + \int_{(c,\pi) \cap 0}^{(\pi+c) \cap 2\pi} 1_{|u-c| \leq \frac{2aN}{N}} |d_N(\varphi_n^i(u) - c)|^p \, du. \quad (33)$$  

But for $N,n$ large enough, if $|u-c| > \frac{2aN}{N}$ we have $|\varphi_n^i(u) - c| > \frac{2}{N}$. Then applying the following simple bound on the Dirichlet kernel,

$$|d_N(u)| \leq \left[ 1 + \frac{\pi}{(2N+1)|u|} \right] \text{ for } |u| \leq \pi,$$

(34)

we get that (33) is lower than

$$\int_{(c,\pi) \cap 0}^{(\pi+c) \cap 2\pi} 1_{|u-c| > \frac{2aN}{N}} \left| \frac{\pi}{(2N+1)(\varphi_n^i(u) - c)} \right|^p \, du + \frac{4a+4}{N}.$$  

Now using $|\varphi_n^i(u) - c| \geq |u-c| - 2a/N$ and applying a simple change of variable, one get the new upper bound $\int_1^\infty w^{-p}dw = \frac{2^{2p}}{2N+1} + \frac{4a+4}{N}$. Then the lemma follows from $\liminf N \rho_n > 0$.  

Lemma 4. Assume $A1$, $A3$ and $A4$ then,

1) $\forall s \in [0,2\pi)$, $\forall \varepsilon > 0$, $\rho_n^{-1} \int_0^{s-\varepsilon} d_{N,n}^{1,2}(s,u)^2 + d_{N,n}^{2,1}(s,u)^2 \, du \overset{N,n \to \infty}{\longrightarrow} 0$,

2) for any continuous function $g : [0,2\pi]^2 \to \mathbb{R}$ the following convergences hold for all $t \in [0,2\pi]$,

$$\begin{align*}
\rho_n^{-1} \int_0^t \int_0^s d_{N,n}^{1,2}(s,u)^2 g(u,s) \, dus \overset{N,n \to \infty}{\longrightarrow} \int_0^t \gamma^{1,2}(s)g(s,s) \, ds, \\
\rho_n^{-1} \int_0^t \int_0^s d_{N,n}^{2,1}(s,u)^2 g(u,s) \, dus \overset{N,n \to \infty}{\longrightarrow} \int_0^t \gamma^{2,1}(s)g(s,s) \, ds, \\
\rho_n^{-1} \int_0^t \int_0^s d_{N,n}^{1,2}(s,u)d_{N,n}^{2,1}(s,u) g(u,s) \, dus \overset{N,n \to \infty}{\longrightarrow} \int_0^t \gamma(s)g(s,s) \, ds.
\end{align*}$$

Proof Using that $g$ is continuous and Lemma 3 with $p = 2$ it is clear that the first property implies the second one. Then, to prove 1) we use that for $u < s-\varepsilon$, $d_{N,n}^{1,i}(s,u)^2 = d_N(\varphi_n^i(s) - \varphi_n^i(u))^2 \leq CN^{-2},$
if $n$ is large enough. Then the result follows from $\liminf N\rho_n > 0$.

**Proof of Theorem 1.** We will use the notations of Proposition 2. Especially, recalling the decomposition (12)–(13) we just have to prove the convergence of $(\rho_n^{-1/2} R_{N,n}^{1,2}(t))_t$ to the process,

$$\frac{1}{2\pi} \int_0^t |h(s)| \sqrt{(\gamma_{1,2}(s) + \tilde{\gamma}_{1,2}(s))\Sigma_{1,1}(s)\Sigma_{2,2}(s) + 2\gamma(s)\Sigma_{1,2}(s)^2} d\tilde{W}_s.$$ 

**First step.** We first prove that the hypothesis $A3$ implies

$$R_{N,n}^{1,2}(t) = M_{N,n}(t) + \tilde{M}_{N,n}(t) + o_P(\rho_n^{1/2}), \quad (35)$$

where we have used the notations (14)–(16).

Using the bound (21) together with Lemma 3 we get,

$$\mathbb{E}[I_{N,n}^1(t)]^2 \leq C \int_0^{2\pi} \left( \int_0^s |d_{N,n}^{1,2}(s,u)|^p du \right)^{2/p} ds \leq C \rho_n^{2/p}, \quad (36)$$

for any $p > 1$. Hence, we have $I_{N,n}^1(t) = o_P(\rho_n^{1/2})$.

Using (19), A1 and Lemma 3, it is clear that $\mathbb{E}[I_{N,n}^3(t)] = o(\rho_n^{1/2})$.

It remains to prove that $\rho_n^{-1} \mathbb{E} I_{N,n}^2(t)^2$ tends to zero as $N, n$ go to infinity in order to obtain (35). This is a little bit more technical. To simplify the notation, we introduce the process

$$Y_{N,n}(t, s) = \int_0^s d_{N,n}^{1,2}(t,u)\sigma^{1,2}(u)dW(u). \quad (37)$$

From Burkholder-Davis-Gundy inequality, $H1$ and Lemma 3 with $p = 2$, it is easy to check that :

$$\left(\mathbb{E} \sup_{s \leq t} |Y_{N,n}(t,s)|^4\right)^{1/4} \leq C(\int_0^t |d_{N,n}^{1,2}(t,u)|^2 du)^{1/2} \leq C \rho_n^{1/2}. \quad (38)$$

With this notation and (18) we have

$$I_{N,n}^2(t)^2 = \frac{1}{4\pi^2} \int_{[0,t]^2} Y_{N,n}(s,s)h_n(s)b^2(s)Y_{N,n}(s',s')h_n(s')b^2(s')dsds'. \quad (39)$$

By symmetry, it is enough to prove that

$$\rho_n^{-1} \mathbb{E} \int_{[0,t]^2} Y_{N,n}(s,s)h_n(s)b^2(s)Y_{N,n}(s',s')h_n(s')b^2(s')1_{s' \leq s} dsds' \to 0. \quad (40)$$

Let $\epsilon > 0$, for $s - \epsilon \leq s' \leq s$ we have using $H1$ and (38)

$$\rho_n^{-1} \mathbb{E} \int_{[0,t]^2} Y_{N,n}(s,s)h_n(s)b^2(s)Y_{N,n}(s',s')h_n(s')b^2(s')1_{s - \epsilon \leq s' \leq s} dsds' \leq C\epsilon. \quad (41)$$
Now if $s' < s - \varepsilon$, we observe that

$$\int_{[0,t]^2} Y_{N,n}(s, s) h_n(s)b^2(s) Y_{N,n}(s', s') h_n(s')b^2(s') 1_{s' < s - \varepsilon} ds ds' = I_{N,n}^{2,1} + I_{N,n}^{2,2},$$

with

$$I_{N,n}^{2,1} = \int_{[0,t]^2} Y_{N,n}(s, s - \varepsilon) h_n(s)b^2(s) Y_{N,n}(s', s') h_n(s')b^2(s') 1_{s' < s - \varepsilon} ds ds',$n

$$I_{N,n}^{2,2} = \int_{[0,t]^2} \int_{s-\varepsilon}^s d_{N,n}^{1,2}(s, u) \sigma^1_s(u) dW(u) h_n(s)b^2(s) Y_{N,n}(s', s') h_n(s')b^2(s') 1_{s' < s - \varepsilon} ds ds'.
$$

From Cauchy-Schwarz inequality, $H1$ and (37)–(38)

$$\rho_n^{-1} \left| \mathbb{E} I_{N,n}^{2,1} \right| \leq C \int_0^t \left( \rho_n^{-1} \int_0^{s-\varepsilon} d_{N,n}^{1,2}(s, u)^2 du \right)^{1/2} ds$$

and from Lemma 4, the right hand side term of the inequality goes to zero. At last we observe by conditioning on $\mathcal{F}_{s-\varepsilon}$ that

$$\mathbb{E} I_{N,n}^{2,2} = \mathbb{E} \int_{[0,t]^2} \int_{s-\varepsilon}^s d_{N,n}^{1,2}(s, u) \sigma^1_s(u) dW(u) h_n(s)b^2(s) - b^2(s-\varepsilon) Y_{N,n}(s', s') h_n(s')b^2(s') 1_{s' < s - \varepsilon} ds ds',$$

and consequently

$$\rho_n^{-1} \left| \mathbb{E} I_{N,n}^{2,2} \right| \leq C(\mathbb{E} \int_0^t |b^2(s) - b^2(s-\varepsilon)|^4 ds)^{1/4}.$$

Finally letting $\varepsilon$ go to zero we obtain the announced result.

Now to study the weak convergence of $\rho_n^{-1/2} R_{N,n}^{1,2}$, we just have to study the limit behavior of the martingale $\rho_n^{-1/2}(M_{N,n} + \tilde{M}_{N,n})$. Following Jacod [11] and Jacod-Protter [13] we just have to determine the limit in probability of the brackets $\langle \rho_n^{-1/2}(M_{N,n} + \tilde{M}_{N,n}), W^r \rangle_t$ for $1 \leq r \leq d$ and $\langle \rho_n^{-1/2}(M_{N,n} + \tilde{M}_{N,n}), \rho_n^{-1/2}(M_{N,n} + \tilde{M}_{N,n}) \rangle_t$, $\forall t \in [0, 2\pi]$.

**Second step.** We prove that for all $r \in \{1, \ldots, d\}$, $\langle \rho_n^{-1/2} M_{N,n}, W^r \rangle_t$ tends to zero in $L^2(\Omega)$ as $N$ goes to infinity. By symmetry the proof will be analogous for $\langle \rho_n^{-1/2} \tilde{M}_{N,n}, W^r \rangle_t$. Using the notation (37) we can write

$$\mathbb{E} \left( \langle \rho_n^{-1/2} M_{N,n}, W^r \rangle_t \right)^2 = \frac{1}{4 \rho_n \pi^2} \int_{[0,t]^2} \mathbb{E} \left( Y_{N,n}(s, s) Y_{N,n}(s', s') \sigma^2_r(s) \sigma^2_r(s') \right) h_n(s) h_n(s') ds ds'.$$

From $H3$, using the duality for stochastic integrals, we have

$$\mathbb{E} \left( Y_{N,n}(s, s) Y_{N,n}(s', s') \sigma^2_r(s) \sigma^2_r(s') \right) = \mathbb{E} \left( \int_0^s d_{N,n}^{1,2}(s, u) \sigma^1_s(u) D_u Y_{N,n}(s', s') \sigma^2_r(s) \sigma^2_r(s') du \right),$$

and consequently

$$\mathbb{E} \left( Y_{N,n}(s, s) Y_{N,n}(s', s') \sigma^2_r(s) \sigma^2_r(s') \right) = E_{N,n}^1(s, s') + E_{N,n}^2(s, s') + E_{N,n}^3(s, s'),$$

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with
\[ E_{N,n}^{1}(s, s') = \mathbb{E}\left( (\sigma^{2,r}(s)\sigma^{2,r}(s')) \int_{0}^{s} d_{N,n}^{1,2}(s, u)d_{N,n}^{1,2}(s', u)1_{\{u \leq s'\}}(\sigma^{1,s'}(u))du \right), \]
\[ E_{N,n}^{2}(s, s') = \mathbb{E}\left( \sigma^{2,r}(s)\sigma^{2,r}(s') \int_{0}^{s} d_{N,n}^{1,2}(s, u)\sigma^{1,s}(u)(\int_{0}^{s'} d_{N,n}^{1,2}(s', v)D_{u}(\sigma^{1,v})dW(v))du \right), \]
\[ E_{N,n}^{3}(s, s') = \mathbb{E}\left( Y_{N,n}(s', s') \int_{0}^{s} d_{N,n}^{1,2}(s, u)\sigma^{1,s}(u)D_{u}(\sigma^{2,r}(s)\sigma^{2,r}(s'))du \right). \]

This leads to the decomposition
\[ \mathbb{E}\left( \rho_{n}^{-1/2}M_{N,n}, W_{t}^{r} \right)^{2} = \frac{1}{4\rho_{n}\pi^{2}} \int_{[0,t]^{2}} (E_{N,n}^{1}(s, s') + E_{N,n}^{2}(s, s') + E_{N,n}^{3}(s, s'))h_{n}(s)h_{n}(s')dsds'. \tag{39} \]

Using successively H1 and Fubini’s theorem, we obtain for the first term
\[ \frac{1}{4\rho_{n}\pi^{2}} \left| \int_{[0,t]^{2}} E_{N,n}^{1}(s, s')h_{n}(s)h_{n}(s')dsds \right| \leq \rho_{n}^{-1} \int_{[0,t]^{2}} \int_{0}^{s} \left| d_{N,n}^{1,2}(s, u) d_{N,n}^{1,2}(s', u)1_{\{u \leq s'\}} \right| dsds', \]
\[ = \rho_{n}^{-1} \int_{0}^{t} \left( \int_{u}^{t} \left| d_{N,n}^{1,2}(s, u) \right| ds \int_{u}^{t} \left| d_{N,n}^{1,2}(s', u) \right| ds \right) du, \]
\[ \leq \rho_{n}^{-1} \int_{0}^{2\pi} \left( \int_{0}^{2\pi} \left| d_{N,n}^{1,2}(s, u) \right| ds \right)^{2} du \leq C\rho_{n}^{-1+2/p}, \]
where we have used Lemma 3 in the last line. Choosing \( p \in (1, 2) \), we deduce that the first term in (39) goes to zero. For the second term, we have from H1, H3 and Cauchy-Schwarz inequality
\[ |E_{N,n}^{2}(s, s')| \leq \int_{0}^{s} \left| d_{N,n}^{1,2}(s, u) \right| |E\sigma^{1,s}(u)| \int_{0}^{s'} \left| d_{N,n}^{1,2}(s', v)D_{u}(\sigma^{1,v})dW(v)\sigma^{2,r}(s)\sigma^{2,r}(s') \right| dvdu, \]
\[ \leq C \int_{0}^{s} \left| d_{N,n}^{1,2}(s, u) \right| du \int_{0}^{s'} \left| d_{N,n}^{1,2}(s', v) \right| dv^{1/2} \leq C\rho_{n}^{1/p+1/2}, \]
this yields
\[ \frac{1}{4\rho_{n}\pi^{2}} \left| \int_{[0,t]^{2}} E_{N,n}^{2}(s, s')h_{n}(s)h_{n}(s')dsds \right| \leq C\rho_{n}^{1/p-1/2} \to 0. \]

We proceed similarly for the third term and this achieves the proof of the second step.

**Third step.** We prove in this section that \( \forall t \in [0, 2\pi] \) the following convergence holds in probability
\[ \left\langle \rho_{n}^{-1/2}M_{N,n}, \rho_{n}^{-1/2}M_{N,n}^{1,2} \right\rangle_{t} \xrightarrow{N,n \to \infty} \frac{1}{4\pi^{2}} \int_{0}^{t} \gamma^{1,2}(s)h(s)^{2}\Sigma^{1,1}(s)\Sigma^{2,2}(s)ds, \]
\[ \left\langle \rho_{n}^{-1/2}M_{N,n}, \rho_{n}^{-1/2}M_{N,n} \right\rangle_{t} \xrightarrow{N,n \to \infty} \frac{1}{4\pi^{2}} \int_{0}^{t} \gamma^{1,2}(s)h(s)^{2}\Sigma^{1,1}(s)\Sigma^{2,2}(s)ds, \]
\[ \left\langle \rho_{n}^{-1/2}M_{N,n}, \rho_{n}^{-1/2}M_{N,n} \right\rangle_{t} \xrightarrow{N,n \to \infty} \frac{1}{4\pi^{2}} \int_{0}^{t} \gamma^{e}(s)h(s)^{2}\Sigma^{1,2}(s)^{2}ds. \]
We only give details for the first convergence. With the preceding notation we have
\[ \left\langle \rho_n^{-1/2} M_{N,n}, \rho_n^{-1/2} M_{N,n} \right\rangle_t = \frac{1}{4\rho_n^2} \int_0^t Y_{N,n}(s, s)^2(\sigma^{2s}\sigma^2(s)h_n(s)^2)ds \]

From Ito’s formula we have
\[ Y_{N,n}(s, s)^2 = \int_0^s d_{N,n}^{1/2}(s, u)^2(\sigma^{1s}\sigma^1)(u)du + 2 \int_0^s Y_{N,n}(s, u)d_{N,n}^{1/2}(s, u)\sigma^{1s}(u)dW(u), \]
and consequently
\[ \left\langle \rho_n^{-1/2} M_{N,n}, \rho_n^{-1/2} M_{N,n} \right\rangle_t = T_{N,n}^1(t) + T_{N,n}^2(t), \]
with
\[
T_{N,n}^1(t) = \frac{1}{4\rho_n^2} \int_0^t \left( \int_0^s d_{N,n}^{1/2}(s, u)^2(\sigma^{1s}\sigma^1)(u)du \right) (\sigma^{2s}\sigma^2(s)h_n(s)^2)ds,
\]
\[
T_{N,n}^2(t) = \frac{1}{2\rho_n^2} \int_0^t \left( \int_0^s Y_{N,n}(s, u)d_{N,n}^{1/2}(s, u)\sigma^{1s}(u)dW(u) \right) (\sigma^{2s}\sigma^2(s)h_n(s)^2)ds.
\]

From Lemma 4 and the continuity of \(\sigma^{s\sigma} = \Sigma^{ij}\) the convergence of \(T_{N,n}^1(t)\) is immediate.

Now, we prove that \(T_{N,n}^2(t)\) tends to zero in \(L^2(\Omega)\). We denote by \(Z_{N,n}(t, s)\) :
\[ Z_{N,n}(t, s) = \int_0^s Y_{N,n}(t, u)d_{N,n}^{1/2}(t, u)\sigma^{1s}(u)dW(u) \]
and we use the notation :
\[ \sigma(s, s')^4 = (\sigma^{2s}\sigma^2)(\sigma^{2s}\sigma^2)(s'). \]

Remark that from \(H1, (38)\) and Lemma 3 with \(p = 2\), we have :
\[ (\mathbb{E}Z_{N,n}(s, s)^2)^{1/2} \leq C\rho_n. \]

We proceed as in the second step, we have :
\[ \mathbb{E}(T_{N,n}^2(t))^2 = \frac{1}{4\rho_n^2} \int_{[0, t]^2} h_n(s)^2h_n(s')^2\mathbb{E} \left( Z_{N,n}(s, s)Z_{N,n}(s', s')\sigma(s, s')^4 \right) dsds'. \]

Now from the duality formula
\[ \mathbb{E} \left( Z_{N,n}(s, s)Z_{N,n}(s', s')\sigma(s, s')^4 \right) = F_{N,n}^1(s, s') + F_{N,n}^2(s, s') + F_{N,n}^3(s, s'), \]
with
\[
F_{N,n}^1(s, s') = \mathbb{E} \left( \sigma(s, s')^4 \int_0^s Y_{N,n}(s, u)d_{N,n}^{1/2}(s, u)(\sigma^{1s}\sigma^1)(u)Y_{N,n}(s', u)d_{N,n}^{1/2}(s', u)1_{\{u \leq s\}}du \right),
\]
\[
F_{N,n}^2(s, s') = \mathbb{E} \left( \sigma(s, s')^4 \int_0^s Y_{N,n}(s, u)d_{N,n}^{1/2}(s, u)\sigma^{1s}(\int_0^{s'} d_{N,n}^{1/2}(s', v)D_u(Y_{N,n}(s', v)\sigma^{1s}(v))dW(v))du \right),
\]
\[
F_{N,n}^3(s, s') = \mathbb{E} \left( Z_{N,n}(s', s') \int_0^s Y_{N,n}(s, u)d_{N,n}^{1/2}(s, u)\sigma^{1s}D_u(\sigma(s, s')^4)du \right).
\]
From $H1$ and (38) we get,
\[
\frac{1}{4\rho_N^2\pi^4} \int_{[0,t]^2} h_n(s)^2 h_n(s')^2 F_{N,n}^1 ds ds' \leq C \rho_n^{-1} \int_{[0,t]^2} \int_0^s |d_{N,n}^{1,2}(s,u)d_{N,n}^{1,2}(s',u)|1_{\{u \leq s'\}} du ds ds',
\]
\[
= C \rho_n^{-1} \int_0^t \left( \int_0^t |d_{N,n}^{1,2}(s,u)| ds \right)^2 du,
\]
and finally from Hölder's inequality and Lemma 3 we obtain for $p > 1$:
\[
\frac{1}{4\rho_N^2\pi^4} \int_{[0,t]^2} h_n(s)^2 h_n(s')^2 F_{N,n}^1 ds ds' \leq C \rho_n^{2/p - 1}.
\]

Turning to $F_{N,n}^3$ we have
\[
\frac{1}{\rho_n^2} \int_{[0,t]^2} h_n(s)^2 h_n(s')^2 F_{N,n}^3 ds ds' \leq C \rho_n^{-2} \int_{[0,t]^2} \int_0^s |d_{N,n}^{1,2}(s,u)||E[Z_{N,n}(s',s')Y_{N,n}(s,u)\sigma^1 D_u(\sigma(s,s')^4)]| du ds ds',
\]
buts, using (43),
\[
E[Z_{N,n}(s',s')Y_{N,n}(s,u)\sigma^1 D_u(\sigma(s,s')^4)] \leq C \rho_n^{1/2}(E[Z_{N,n}(s',s')]^2)^{1/2} \leq C \rho_n^{3/2},
\]
consequently for any $p > 1$
\[
\frac{1}{4\rho_N^2\pi^4} \int_{[0,t]^2} h_n^2(s)^2 h_n^2(s')^2 F_{N,n}^3 ds ds' \leq C \rho_n^{1/p - 1/2}.
\]

We bound the last term on a similar way observing that
\[
D_u(Y_{N,n}(s',v)\sigma^1(v)) = Y_{N,n}(s',v)D_u(\sigma^1(v)) + d_{N,n}^{1,2}(s',u)\sigma^1(u)1_{\{u \leq v\}} \sigma^1(v)
+ \int_0^v d_{N,n}^{1,2}(s',v')D_u(\sigma(v'))dW_v \sigma^1(v).
\]

Finally we conclude that $E(T_{N,n}^2(t))^2$ tends to zero as $N,n$ go to infinity.

**Fourth step.** We turn back to the decomposition of $R_{N,n}^{1,2}(t)$ given in (35). We deduce from the second step that \( \rho_n^{-1/2} (M_{N,n} + \tilde{M}_{N,n}, W^r) \) tends to zero, $\forall r \in \{1, \ldots, d\}$. And from the third step we can prove that
\[
\lim_{N,n} (\rho_n N_{N,n}, \rho_n N_{N,n})_t = \frac{1}{4\pi^2} \int_0^t h(s)^2 \left[ (\gamma^{1,2}(s) + \zeta^{1,2}(s))\Sigma^{1,1}(s)\Sigma^{2,2}(s) + 2\gamma^c(s)\Sigma^{1,2}(s)^2 \right] ds
\]
where $N_{N,n}(t) = M_{N,n}(t) + \tilde{M}_{N,n}(t)$. This achieves the proof of Theorem 1.

5.2. Proof of Proposition 1

Using the decomposition (6)–(7) the proof follows from the same lines as the proof of Theorem 1 where we replace $d_{N,n}^{1,2}(s,u)$ by $d_N(s-u)$. The main differences are, that we replace Lemma 3 by the simple inequality $\int_0^\pi |d_N(u)|^p \leq C_p N^{-1}$, and in step 3, Lemma 4 is replaced by the property of the Dirichlet kernel,
\[
N \int_0^t \int_0^s d_N(s-u)^2 g(u,s)du ds \xrightarrow{N \to \infty} \frac{\pi}{2} \int_0^t g(s,s)ds.
\]
5.3. Proof of Lemma 1

i) We first show the following convergence

$$\forall 0 < s < 2\pi, \lim_{N,n \to \infty} n \int_0^{\varphi_n(s)+2\pi/n} d_N(\varphi_n(u))^2 du = 2\pi(\eta(2a) + 1). \quad (44)$$

For $0 < s < 2\pi$, we note $k_n(s) = n\varphi(s)/2\pi$. We have

$$\int_0^{\varphi_n(s)+2\pi/n} d_N(\varphi_n(u))^2 du = \frac{2\pi}{n} \sum_{k=0}^{k_n(s)} d_N\left(\frac{2k\pi}{n}\right)^2$$

But

$$\sum_{k=0}^{k_n(s)} d_N\left(\frac{2k\pi}{n}\right)^2 = 1 + \sum_{k=1}^{\infty} \frac{1}{(2N+1)^2} \frac{\sin^2((2N+1)k\pi/n)}{\sin^2(k\pi/n)} 1_{\{k \leq k_n(s)\}}.$$}

We have

$$\lim_{N,n} \frac{1}{(2N+1)^2} \frac{\sin^2((2N+1)k\pi/n)}{\sin^2(k\pi/n)} = \frac{\sin^2(2a\pi k)}{(2a\pi k)^2},$$

and since $k_n(s)\pi/n \leq s/2 < \pi$, using $\sin x \geq Cx$ for $x \leq s/2 < \pi$,

$$\frac{1}{(2N+1)^2} \frac{\sin^2((2N+1)k\pi/n)}{\sin^2(k\pi/n)} 1_{\{k \leq k_n(s)\}} \leq \frac{C}{(2a)^2}.$$}

We conclude then from the dominated convergence Theorem that

$$\lim_{N,n} \sum_{k=0}^{k_n(s)} d_N\left(\frac{2k\pi}{n}\right)^2 = 1 + \sum_{k=1}^{\infty} \frac{\sin^2(2a\pi k)}{(2a\pi k)^2},$$

and if we note

$$\eta(a) = \sum_{k=1}^{\infty} \frac{\sin^2(a\pi k)}{(a\pi k)^2},$$

we obtain

$$\lim_{N,n} n \int_0^{\varphi_n(s)+2\pi/n} d_N(\varphi_n(u))^2 du = 2\pi(\eta(2a) + 1).$$

ii) We now prove the explicit expression for $\eta(a)$ given in (26). Defining $f(x) = x(1-x)$ on $[0, 1]$, we have the Fourier Development

$$f(x) = \frac{1}{6} - \sum_{k=1}^{\infty} \frac{\cos(2\pi kx)}{\pi^2 k^2},$$

we deduce then that

$$\sum_{k \geq 1} \frac{\cos(2\pi k a)}{k^2} = \sum_{k \geq 1} \frac{\cos(2\pi k(a - [a]))}{k^2} = \pi^2 \left(\frac{1}{6} - r(a)(1-r(a))\right),$$

where $r(a) = \frac{1}{2} - \frac{a}{\pi}$.
with \( r(a) = a - [a] \). Turning back to \( \eta(a) \), we obtain

\[
\eta(a) = \sum_{k=1}^{\infty} \frac{1 - \cos(2\pi k a)}{(2\pi^2 a^2 k^2)},
\]

\[
= \frac{1}{2a^2} r(a)(1 - r(a)).
\]

iii) Finally, we show (25). For \( s < 2\pi \) we write

\[
\int_{0}^{s} d_N(\varphi_n(s) - \varphi_n(u))^2 du = V_{N,n}^1(s) + V_{N,n}^2(s),
\]

with

\[
V_{N,n}^1(s) = \int_{\varphi_n(s)}^{s} d_N(\varphi_n(s) - \varphi_n(u))^2 du,
\]

\[
V_{N,n}^2(s) = \int_{0}^{\varphi_n(s)} d_N(\varphi_n(s) - \varphi_n(u))^2 du.
\]

Since \( d_N(0) = 0 \), we have \( V_{N,n}^1(s) = s - \varphi_n(s) \), and we deduce

\[
n \int_{0}^{t} V_{N,n}^1(s) ds = n \sum_{k=0}^{k_n(t)-1} \int_{2\pi k/n}^{2\pi (k+1)/n} (s - 2\pi k/n) ds + o(1/n) \rightarrow t\pi,
\]

where we used the notation \( k_n(t) = \lfloor nt \rfloor \).

Now we set \( v = \varphi_n(s) + 2\pi/n - u \) in \( V_{N,n}^2(s) \) and we obtain for \( 0 < s < 2\pi \):

\[
n V_{N,n}^2(s) = n \int_{2\pi/n}^{\varphi_n(s)+2\pi/n} d^2_N(\varphi_n(v)) dv
\]

\[
= n \int_{0}^{\varphi_n(s)+2\pi/n} d^2_N(\varphi_n(v)) dv - 2\pi.
\]

From (44), we deduce that

\[
n \int_{0}^{t} V_{N,n}^2(s) ds \rightarrow t2\pi \eta(2a).
\]

The result follows from (45)–(47).

5.4. Proof of Lemma 2

i) Let \( s \in [0, t] \), we have

\[
n \int_{0}^{s} d_{N,n}^{1,2}(s, u)^2 du = n \int_{\varphi_n^1(s)}^{s} d_{N,n}^{1,2}(s, u)^2 du + n \int_{0}^{\varphi_n^1(s)} d_{N,n}^{1,2}(s, u)^2 du
\]

\[
= V_{N,n}^1(s) + V_{N,n}^2(s).
\]

Now if \( u \in [\varphi_n^1(s), s] \), \( |\varphi_n^1(s) - \varphi_n^2(s)| = \pi/n \) and then \( d_{N,n}^{1,2}(s, u) = d_N(\pi/n) \). This gives

\[
V_{N,n}^1(s) = nd_N(\pi/n)^2(s - \varphi_n^1(s)),
\]

\[
V_{N,n}^2(s) = nd_N(\pi/n)^2(s - \varphi_n^1(s)),
\]

\[
V_{N,n}^3(s) = nd_N(\pi/n)^2(s - \varphi_n^1(s)).
\]
and finally
\[ \lim_{N,n} \int_0^t V_{N,n}^1(s) ds = \pi \frac{\sin^2(a\pi)}{(a\pi)^2} t. \]

To compute the limit of \( V_{N,n}^2(s) \), we make the change of variable \( v = \varphi^1_n(s) + 2\pi/n - u \). One can easily check that \( \varphi^2_n(u + 2k\pi/n) = \varphi^2_n(u) + 2k\pi/n \) and that \( \varphi^2_n(-u) = -\varphi^2_n(u) - 2\pi/n \), du a.e., if \( |u| \geq \pi/n \).

This leads to
\[ V_{N,n}^2(s) = n \int_{2\pi/n}^{\varphi^1_n(s)+2\pi/n} d_N(\varphi^2_n(v))^2 dv. \]

Now since \( \varphi^1_n(s) + 2\pi/n \leq t < 2\pi \), we have for all \( \varepsilon > 0 \):
\[ \lim_{N,n} \int_{\varepsilon}^{\varphi^1_n(s)+2\pi/n} d_N(\varphi^2_n(v))^2 dv = 0. \]

But we can establish for \( 0 < \varepsilon < \pi \)
\[ \lim_{N,n} \int_{2\pi/n}^{\varepsilon} d_N(\varphi^2_n(v))^2 dv = \pi \left( 2(\eta(a) - \eta(2a)) - \frac{\sin^2(a\pi)}{(a\pi)^2} \right), \]

it yields that
\[ \lim_{N,n} V_{N,n}^2(s) = \pi \left( 2(\eta(a) - \eta(2a)) - \frac{\sin^2(a\pi)}{(a\pi)^2} \right), \]

and finally
\[ \lim_{N,n} \int_0^t (V_{N,n}^1(s) + V_{N,n}^2(s)) ds = 2\pi(\eta(a) - \eta(2a))t = \gamma_1(a)t. \]

It remains to establish (48) to finish the proof of i). Let \( \varphi^2_n(\varepsilon) = (2k\varepsilon + 1)\pi/n \). We have
\[ \lim_{N,n} \int_{2\pi/n}^{\varepsilon} d_N(\varphi^2_n(v))^2 dv = \lim_{N,n} \int_{2\pi/n}^{\varphi^2_n(\varepsilon)-\pi/n} d_N(\varphi^2_n(v))^2 dv = \lim_{N,n} \int_{2\pi/n}^{\varphi^2_n(\varepsilon)-2\pi/k/n} d_N(\varphi^2_n(v))^2 dv, \]

but
\[ n \int_{2\pi/n}^{\varphi^2_n(\varepsilon)-\pi/n} d_N(\varphi^2_n(v))^2 dv = n \sum_{k=1}^{k_1-1} \int_{2\pi/k/n}^{2\pi(k+1)/n} d_N(\varphi^2_n(v))^2 dv \]
\[ = n\pi/n \sum_{k=1}^{k_1-1} \left( d_N((2k - 1)\pi/n)^2 + d_N((2k + 1)\pi/n)^2 \right). \]

By dominated convergence, we have
\[ \lim_{N,n} \sum_{k=1}^{k_1-1} \left( d_N((2k - 1)\pi/n)^2 + d_N((2k + 1)\pi/n)^2 \right) = \sum_{k=1} \left( \frac{\sin^2(a\pi(2k - 1))}{(a\pi(2k - 1))^2} + \frac{\sin^2(a\pi(2k + 1))}{(a\pi(2k + 1))^2} \right). \]

Now recalling that \( \eta(a) = \sum_{k=1} \frac{\sin^2(a\pi(k))}{(a\pi k)^2} \) we deduce
\[ \sum_{k=1} \frac{\sin^2(a\pi(2k - 1))}{(a\pi(2k - 1))^2} = \eta(a) - \eta(2a), \]

and consequently (48) is proved.
We prove ii) on the same way by introducing the decomposition
\[
n \int_0^s \, d_{N,n}^{2,1}(s, u)^2 du = n \int_0^s \, d_{N,n}^{2,1}(s, u)^2 du + n \int_0^\phi_2(s) \, d_{N,n}^{2,1}(s, u)^2 du
\]
\[
= V_{N,n}^1(s) + V_{N,n}^2(s).
\]
As previously it is easy to see that
\[
\lim_{N,n} \int_0^t V_{N,n}^1(s) ds = \pi \frac{\sin^2(a\pi)}{(a\pi)^2} t.
\]
To compute the limit of \(V_{N,n}^2(s)\), we set \(v = \phi_n^2(s) + \pi/n - u\). We observe that \(\phi_n^1(u + 2k\pi/n) = \phi_n^1(u) + 2k\pi/n\) and that \(\phi_n^1(-u) = -\phi_n^1(u) - 2\pi/n\), du a.e. This gives
\[
V_{N,n}^2(s) = n \int_{\pi/n}^{\phi_n^2(s) + \pi/n} d_N(\phi_n^1(v) + \pi/n)^2 dv.
\]
We conclude as in i) remarking that for \(0 < \varepsilon < \pi\)
\[
\lim_{N,n} \int_{\pi/n}^\varepsilon \, d_N(\phi_n^1(v) + \pi/n)^2 dv = \pi \left(2(\eta(a) - \eta(2a)) - \frac{\sin^2(a\pi)}{(a\pi)^2}\right).
\]
We turn now to iii). We have
\[
n \int_0^s \, d_{N,n}^{1,2}(s, u)d_{N}^{2,1}(s, u) du = n \int_0^s \, d_{N,n}^{1,2}(s, u)d_{N}^{2,1}(s, u) du
\]
\[
+ n \int_0^{\phi_n^1(s)} \, d_{N,n}^{1,2}(s, u)d_{N}^{2,1}(s, u) du,
\]
\[
= V_{N,n}^1(s) + V_{N,n}^2(s).
\]
As in i) and ii)
\[
\lim_{N,n} \int_0^t V_{N,n}^1(s) ds = \pi \frac{\sin^2(a\pi)}{(a\pi)^2} t.
\]
It remains to identify \(\lim_{N,n} V_{N,n}^2(s)\). Let \(v = \phi_n^1(s) + 2\pi/n - u\), we have
\[
V_{N,n}^2(s) = n \int_{2\pi/n}^{\phi_n^1(s) + 2\pi/n} d_N(\phi_n^1(v) + \phi_n^2(s) - \phi_n^1(s))d_N(\phi_n^2(v)) dv,
\]
Now for \(0 < \varepsilon < \pi\) and \(\phi_n^1(\varepsilon) = 2\pi k\varepsilon/n\),
\[
\lim_{N,n} \int_{2\pi/n}^{\varepsilon} \, d_N(\phi_n^1(v) + \phi_n^2(s) - \phi_n^1(s))d_N(\phi_n^2(v)) dv = \pi \lim_{N,n} \sum_{k=1}^{k_n} d_N((2k + \delta_n(s))\pi/n) (d_N((2k - 1)\pi/n)
\]
\[+ d_N((2k + 1)\pi/n)),
\]
(50)
where \(\delta_n(s) = 1\) if \(2k\pi/n + \pi/n \leq s < 2(k+1)\pi/n\) and \(\delta_n(s) = -1\) otherwise. We deduce then that
\[
\lim_{N,n} \int_0^t V_{N,n}^2(s) ds = \pi 2 \sum_{k \geq 1} \left(\frac{\sin((2k + 1)\pi a)}{(2k + 1)\pi a} + \frac{\sin((2k - 1)\pi a)}{(2k - 1)\pi a}\right)^2 t.
\]
(51)
A tedious calculation gives

\[ \sum_{k \geq 1} \frac{\sin((2k + 1)\pi a)}{(2k + 1)\pi a} \cdot \frac{\sin((2k - 1)\pi a)}{(2k - 1)\pi a} = -\frac{1}{2} \frac{\sin^2(\pi a)}{(\pi a)^2} + \frac{\sin(2\pi a)}{8\pi^2 a^2} \left( 1_{(0,1/2)}(r(a)) - 1_{(1/2,1)}(r(a)) \right), \quad (52) \]

and putting this together we obtain

\[ \lim_{N,n} \int_0^t V_{N,n}^2(s) ds = \pi \left( (\eta(a) - \eta(2a)) - \frac{\sin^2(\pi a)}{(\pi a)^2} + \frac{\sin(2\pi a)}{8\pi^2 a^2} \left( 1_{(0,1/2)}(r(a)) - 1_{(1/2,1)}(r(a)) \right) \right) t, \]

and finally

\[ \lim_{N,n} \int_0^t (V_{N,n}^1(s) + V_{N,n}^2(s)) ds = \pi \left( (\eta(a) - \eta(2a)) + \frac{\sin(2\pi a)}{8a^2} \left( 1_{(0,1/2)}(r(a)) - 1_{(1/2,1)}(r(a)) \right) \right) t = \gamma_2(a)t. \]

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Figure 1: Mean (solid line) and standard deviation (dashed line) of $\Gamma_{N,N}^{1,1}$. 

Figure 2: Variance of $\Gamma_{N,n}^{1,1}$ multiplied by $n$ as a function of $N$. 
Figure 3: Plot of the function $v$

Figure 4: Mean (solid line) and standard deviation (dashed line) of $\Gamma_{N,n}^{\text{unbiased}}$ as a function of $N$. Non synchronous case.
Figure 5: Variance of $\Gamma_{N,n}^{\text{unbiased}}$ multiplied by $n$ as a function of $N$. Non synchronous case.