American put option on a defaultable asset and option based portfolio insurance

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Abstract

We consider a market with a defaultable asset and maximize a CRRA utility function of the terminal wealth of a portfolio with the constraint that the wealth at any time should be greater than a fixed floor $K$. The optimal strategy is built thanks to an American put option on a defaultable asset as in [8], therefore we precisely study this kind of option. We also give an extension of the early exercise premium formula to this setting. It will be useful to get numerical approximations of the option value and its behavior when the default intensity goes to 0.

Keywords: American option, Defaultable asset, Incomplete market, CRRA, OBPI.

1 Introduction

Since the subprime crisis, credit derivatives have been under increased scrutiny. Credit Default Swap and Collateralized Debt Obligation have been identified as 'toxic' due to counterpart risk, lack of transparency and inefficient trade processing. Therefore investors appreciate to deal with portfolios based on equity options and a promised guarantee allows them to manage credit risk even if the credit derivatives market is frozen. This kind of portfolios is related to the concept of Option Based Portfolio Insurance (OBPI) introduced by Leland and Rubinstein (see [14]). Assume that we can buy or sell assets as DZC (Default-Zero-Coupon) bonds in the market. In order to avoid large losses, we ensure a minimum value at any time and maximize the final wealth with a risk aversion given by a utility function $U$. The latter could be formulated as:

$$
\max_{\pi \in \mathcal{A}} \mathbb{E} \left[ U(V_T^{x,\pi}) \right], \quad V_t^{x,\pi} \geq K, \quad \forall t \in [0, T].
$$
where $x$ is the initial wealth and $\mathcal{A}$ the class of admissible portfolios which will be defined later. Black and Jones (see \[3\]) and Perold (see \[19\]) also developed the Constant Proportion Portfolio Insurance (CPPI) method which has become popular among practitioners. As it improves the return and the risk exposure, OBPI is used by many hedge funds. This kind of strategy is also useful to hedge the protection promised by a Credit Default Swap. As the underlying assets are liquid, OBPI on DZC of the firm mentioned on the CDS is a viable hedging instrument.

In order to solve the OBPI problem, we use the methodology developed in \[8\]. In this approach, authors work in the Black and Scholes framework with an underlying asset which is not submitted to the default risk. In our case, it follows from the default risk that the market is incomplete and option values should be specified. The defaultable asset $\bar{S}$ can be described by a classical asset which pays dividends until the default time and which jumps proportionally to a recovery rate $R \in [0,1]$ at this time. Meanwhile, all parameters of the asset (drift, volatility) change after the default since it impacts the risk aversion of buyers and sellers.

We work in a hazard rate model, we consider $\tau$ the default time and $\gamma = (\gamma_t)_{t \geq 0}$ the intensity process, then the asset $\bar{S}$ pays $\bar{S}_t - R\gamma_t dt$ on $(t,t+dt)$ and if the jump occurs, the asset jumps proportionally to $-R$. If the default does not occur, investors receive dividends else they lose money because of the jump. To understand the impact of the default on this framework, we define a reference filtration $\mathbb{F}$ generated by a Brownian motion that we enlarge progressively with the occurrence of the default. We call the new filtration $\mathbb{G}$, where $\mathcal{G}_t = \bigcap_{\epsilon > 0} \mathcal{F}_{t+\epsilon} \vee \sigma(\tau \wedge t+\epsilon)$. Assuming the default time is independent of Brownian motion, the hypothesis (\(H\)) holds and the $\mathbb{F}$-intensity process can be assumed deterministic. Many authors worked on the dynamics of defaultable assets as CDS or defaultable bond (see \[3\]). Thanks to a representation theorem, they wrote the dynamics of such asset in this framework. The new part of the representation is the martingale due to default that we shall describe later. The main aim of our paper is to understand the default risk to define the optimal OBPI strategy in a market with defaultable assets.

We shall construct a portfolio such that its wealth at any time is greater than a fixed floor $K$. This methodology is based on the inequality $V^\pi_t := \pi S_t + P^\pi_K(t, \pi S_t) \geq K$ (where $\pi$ is constant, $P^\pi_K$ the price of American put written on $\pi S$ with a strike $K$). Unfortunately the process $V^\pi$ is not a martingale and the corresponding strategy is not self-financing. To solve this problem, authors in \[8\] define a process $\pi$ such that $V^\pi$ is a martingale, greater than the floor $K$ and optimal at $T$ considering the risk aversion defined by a CRRA utility function. In our framework, considering the default, the problem is more difficult since the market is incomplete (due to the default) and all dynamics jumps. To solve the problem we have to choose a fixed risk neutral probability measure. Using the dual theory and some results of \[11\], we shall characterize the optimal risk neutral probability measure $\mathbb{P}^*$ and prove that coefficients of the Radon Nikodym density of $\mathbb{P}^*$ with respect to $\mathbb{P}$ on $\mathcal{G}_T$ are deterministic functions of time and default occurrence. Moreover, the optimal OBPI strategy implies an American put option and we have to improve our knowledge on American option written on defaultable assets.

American put options have been widely studied in jump diffusion model (see \[20\], \[21\], \[22\]). However, in our setting this is a new challenge as all asset’s parameters (drift, volatility) change with the occurrence of the default. Moreover, dealing with only one jump is more difficult since the $\mathcal{G}$-intensity
process cannot be deterministic (since this process is null after the default) in the large filtration in the opposite of the standard jump model where the intensity process of jumps is deterministic. We shall describe the American put option prices before and after the jump and characterize them as a solution of a system of variational inequalities. Thanks to this property, we improve our understanding on how the default impact this option. Especially, we extend the early exercise premium formula to our setting and this allows us to write the price of the American put option as the price of the European put option on non defaultable asset plus an extra term called the swap default premium. The characterization of the American put option price and the description of its exercise region leads us to construct the optimal strategy which solves the OBPI problem. We also derive some asymptotic results when the intensity process goes to zero. In the case of European option, some asymptotic results have been given in [1].

The paper is organized in four parts. The first part is dedicated to set the framework. We give the dynamic of the risky asset, specify the maximization problem related to the OBPI and determine the risk-neutral probability measure under which we shall work. In the second part, we study the price of an American put option on a defaultable asset. We give some analytical properties of its value function, describe its exercise region and emphasize on the difficulty due to default event. In the third part, we solve the OBPI problem. In the last part, we extend the early exercise premium formula to our setting and use it to get numerical results. Finally, we study the asymptotic of the swap premium default when the intensity process goes to zero.

2 The Model and the financial market

2.1 The model

We consider a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ on which are defined a Brownian motion $B$ and a positive random time $\tau$ which represents the default time of one firm. The filtration $\mathcal{G} = (\mathcal{G}_t)_{t \geq 0}$ is right-continuous and defined by $\mathcal{G}_t = \bigcap_{\epsilon > 0} \mathcal{F}_{t+\epsilon} \vee \sigma(\tau \wedge t + \epsilon)$ where $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ is the natural filtration generated by the Brownian motion $B$. We work in a hazard model (see [3] for more details): for $t \geq 0$, we set $H_t = 1_{\tau \leq t}$ and assume that there exists a positive constant $\lambda$ such that, the process $\tilde{N}$ defined by $\tilde{N}_t = H_t - \lambda(t \wedge \tau)$ is a $\mathcal{G}$-martingale. Hence, for $t \geq 0$, we have $\mathbb{P}(\tau \geq t) = \exp(-\lambda t)$. We also assume that the Brownian motion is independent of $N$ and it follows that $\tau$ is a $\mathcal{G}$- stopping time totally inaccessible. In this framework, it is obvious that the hypothesis $(\mathcal{H})$ is satisfied. We recall that it implies that any $\mathcal{G}-$ martingale $M$ can be decomposed as follows $M = M_0 + Z.B + U.N$ where the processes $Z$ and $U$ are $\mathcal{G}-$ predictable.

2.2 The financial market

We consider a financial market composed of two assets: the money market $S^0$ solution of $dS^0_t = rS^0_t dt$ with $r > 0$ and the defaultable asset $\bar{S}$ which satisfies the following differential equation:

$$d\bar{S}_t = \bar{S}_t \left[ \mu(H_t)dt + \sigma(H_t)dB_t - RdN_t \right],$$
where $R \in [0, 1]$ is the recovery rate and $(\mu(0), \mu(1), \sigma(0), \sigma(1)) \in \mathbb{R}^2 \times (0, +\infty)^2$. In order to emphasize on the markovianity of the model, we remark that $\bar{S}_t = (1 - R)H_tS_t$ where the dynamic of the process $S$ is
\[
dS_t = S_t [(\mu(H_t) + R\gamma(H_t))dt + \sigma(H_t)dB_t], \text{ with } \gamma(h) = (1 - h)\lambda \text{ for } h \in \{0, 1\}.
\]
(1)

It is easy to see that
\[
S_t = S_0 \exp \left( \int_0^t \left( \mu(H_s) - \frac{\sigma(H_s)^2}{2} + R\gamma(H_s) \right) ds + \int_0^t \sigma(H_s) dB_s \right).
\]

Therefore, the process $(S, H)$ is Markovian. The process $S$ is called the predefault price process. To insure the market viability, the possible jump of $S$ is compensated by dividends, according to the dynamic of the process $S$. Investors buy defaultable assets because they hope that the default event will not happened but they should deal with a new risk: "the default risk".

We consider a new parameter $h \in \{0, 1\}$ which define the occurrence of the default. If the parameter $h = 0$ then the default has not occurred else $h = 1$. The process $(\gamma(H_t))_{0 \leq t \leq T}$ is the intensity process under the historical probability $\mathbb{P}$. At the opposite to classical models with jumps, the intensity jumps to 0 at the default time and, after the default ($h = 1$), the market follows the Black and Scholes model with new drift and volatility parameters. Therefore, the market is complete after the default. The main difficulty is to deal before the default event since it is assumed totally inaccessible.

Let $\beta$ and $\delta$ two $\mathcal{G}$-predictable processes, such that $\delta$ takes values in $(-1, +\infty)$. We denote by $Z^{\delta, \beta}$ the solution of the following equation:
\[
dZ_t^{\delta, \beta} = Z_t^{\delta, \beta} [\beta_t dB_t + \delta_t dN_t], \quad Z_0^{\delta, \beta} = 1,
\]

Applying Ito’s formula, we get:
\[
d(e^{-rt}S_tZ_t^{\delta, \beta}) = e^{-rt}S_tZ_t^{\delta, \beta} [\beta_t dB_t + (\delta_t - R(1 + \delta_t))dN_t + \alpha(t, H_t)dt],
\]
where we have set
\[
\alpha(t, H_t) = \mu(H_t) - r + \sigma(H_t)\beta_t - R\delta_t\gamma(H_t)).
\]

Therefore, if the processes $\beta$ and $\delta$ satisfy the following equation:
\[
\mu(H_t) - r + \sigma(H_t)\beta_t - R\delta_t\gamma(H_t)) = 0, \quad \delta_t > -1,
\]
(2)
the process $Z_t^{\delta, \beta}\bar{S}/S^0$ is a local martingale under the probability $\mathbb{P}^\delta$ such that $Z_T^{\delta, \beta}$ is the Radon-Nikodym density of $\mathbb{P}^\delta$ with respect to $\mathbb{P}$ on $\mathcal{G}_T$. When the processes $\beta$ and $\delta$ satisfy equation (2), we use the notation $Z^{\delta} := Z^{\delta, \beta}$. Since $\gamma(1) = 0$, we have $\beta_t = (r - \mu(1))/\sigma(1)$ on $(\tau \leq t)$ for every risk neutral probability, however we can construct an infinite number of risk neutral probabilities.

### 2.3 Choice of the risk neutral probability

As it exists many risk neutral probabilities, we shall determine the risk neutral probability chosen by the investor. We recall that we will consider investor with a CRRA utility function $U$ defined
by $U(x) = x^{1-\alpha}/(1 - \alpha)$ with $\alpha \in (0, 1)$. Hence he will choose an optimal risk neutral probability with respect to his utility function. To identify the optimal risk neutral probability, let recall the unconstrained optimization problem and its dual representation given by Delbaen and Schachermayer (see [6]). We set

$$A(x) = \{(V_t)_{t \geq 0} \in \mathbb{G}\text{ – adapted} : V_0 = x \text{ and } (V_t)_{t \geq 0} \text{ is the value process of a self-financing portfolio}\}.$$ 

The unconstrained problem is the following

$$u(x) = \sup_{(V_t)_{t \geq 0} \in A(x)} \mathbb{E} [U(V_T)] , \quad \forall x > 0$$

where $x$ represents the initial wealth, $T$ the horizon time investment. To solve this problem, authors in [6] define the dual function of $u$:

$$v(y) = \sup_{x > 0} \{u(x) - xy\}, \quad \forall y > 0$$

They prove that the dual function of $u$ is given by:

$$v(y) = \min_{\delta \in \mathbb{Q}} \left[ \tilde{U}(ye^{-rT}Z_T^\delta) \right], \quad \forall y > 0$$

where $\tilde{U}$ is the dual function of $U$ and $\mathbb{Q}$ is the set of $\mathbb{G}$-predictable processes $\delta$ with values in $(-1, +\infty)$ such that $Z_T^\delta$ is well defined. Proving the existence of an optimal risk neutral probability $\mathbb{P}^{\delta^*}$ with $\delta^* \in \mathbb{Q}$ such that the minimum of the dual function is reached, they showed the existence of the optimal wealth and gave an explicit characterization:

$$\delta^* = \arg\min \left[ \tilde{U}(ye^{-rT}Z_T^\delta) \right] \quad \text{and} \quad V_T^* = I \left[ ye^{-rT}Z_T^* \right],$$

where the function $I$ is the inverse function of $U'$ and $y = u'(x)$.

In the constrained optimization problem, we will construct (see section 4) the optimal wealth thanks to the one found in the unconstrained case. Therefore, we shall work under the optimal risk neutral probability $\mathbb{P}^* := \mathbb{P}^{\delta^*}$.

In the next Proposition, we prove that $\delta^*$ can be described by a closed formula, especially, it is proved that it is a deterministic function of the default event.

**Proposition 1.** We denote by $x^*$ the unique solution of the following equation on $(-1, +\infty)$.

$$(\kappa + 1)\lambda \frac{R^2}{\sigma(0)^2} x - (x + 1)^{-(\kappa + 1)} = (\kappa + 1) \frac{R}{\sigma^2(0)} (\mu(0) - r) - 1,$$

where $\kappa = \frac{1 - \alpha}{\alpha}$.

We equally set

$$\delta^*_t := x^* \mathbb{1}_{\{H_t = 0\}} \quad \text{and} \quad \beta^*_t = \frac{1}{\sigma(H_t)} (r - \mu(H_t) + Rx^* \gamma(H_t)).$$

The process $Z^*$, solution of $dZ^*_t = Z^*_t \left[ \beta^*_t dB_t + \delta^*_t dN_t \right]$, $Z^*_0 = 1$, is the density of the optimal risk neutral probability measure $\mathbb{P}^*$ with respect to $\mathbb{P}$.  


Proof: see the Appendix.

Notice that the processes $\beta^*$ and $\delta^*$ satisfy equation (2), especially we have $\beta^*_t \mathbb{I}_{(H_t=1)} = (r - \mu(1))/\sigma(1)$. For $t \in [0, T]$, we set $\lambda^* = (1 + x^*)\lambda$ and $\gamma^*(H_t) = (1 - H_t)\lambda^*$, then under $\mathbb{P}^*$, the process $\bar{S}$ and $S$ are respectively solutions of the following stochastic differential equations:

\[
\begin{align*}
\bar{S}_t &= \bar{S}_t \left[ rdt + \sigma(H_t)dB^*_t - RdN^*_t \right], \\
S_t &= S_t \left[ (r + R\gamma^*(H_t))dt + \sigma(H_t)dB^*_t \right],
\end{align*}
\]

where the process $B^*$ defined by $B^*_t = B_t - \int_0^t \beta^*_s ds$ is a Brownian motion under $\mathbb{P}^*$ and the process $N^*$, defined by $N^*_t = H_t - \lambda^*(t \wedge \tau)$ is a $\mathbb{P}^*$-martingale. Throughout the paper, we will work under $\mathbb{P}^*$ the optimal risk neutral probability for a CRRA investor defined in Proposition 1 and denote by $\mathbb{E}^*$ the expectation under $\mathbb{P}^*$.

### 3 Option Based Portfolio Insurance

We construct an option based portfolio insurance (OBPI) on a defaultable asset, using American put options. We shall work with a CRRA (constant risk relative aversion) utility function defined on $\mathbb{R}^+$ by $U(x) = x^{1-\alpha}/(1 - \alpha)$ for $\alpha \in (0, 1)$. We assume that the initial wealth is $x \geq K$ where $K > 0$ and notice that the terminal wealth of a portfolio $V_T$ is a random variable, measurable with respect to $\mathcal{G}_T$.

Our goal is to solve the following problem:

\[
\max_{(V_t)_{t \geq 0}} \mathbb{E} \left[ U(V_T) \right],
\]

with the constraints $V_0 = x$, $\forall t \in [0, T]$, $V_t \geq K$, and $(e^{-rt}V_t)_{t \geq 0}$ is a $\mathbb{P}^*$-martingale. The martingale constraint imposes to consider only self financing processes. We remark that to guarantee the floor $K$ at any time, an obvious strategy consists in buying an asset and an American put option with strike $K$ on this asset. We would have $V_t = \bar{S}_t + P^a_t \geq K$. However, with this strategy, the actualised wealth process is not a $\mathbb{P}^*$-martingale and the terminal wealth does not maximize the utility function at time $T$. To solve the constrained optimization problem, we shall follow the approach of Lacoste, Jeanblanc and El Karoui (see [8]) and find a $\mathbb{G}$-adapted process $(X_t)_{t \geq 0}$ such that $V_t = X_t + \bar{P}_t \geq K$ satisfies the martingale constraint where $\bar{P}_t$ is the price of an American put option on $X$. It implies to have a good knowledge of American put option on defaultable assets.

#### 3.1 American put option on a defaultable asset

In this section, we study an American put option on the defaultable asset. First, using the fact that the process $(S, H)$ is Markovian, we define its price function. From the Snell envelope theory, we deduce a characterization of the optimal stopping time and introduce exercise and continuation regions. Finally, we characterize the option value function as the unique solution of a free boundary
American put option and OBPI

problem. An important fact is that the free boundary will jump at the default time. Some estimations of the free boundary and numerical approximations of it will be equally provided, thanks to the early exercise formula in Section 1.

Since we work in an incomplete market and will consider CRRA investors, we choose an optimal risk neutral probability $\mathbb{P}^*$ as defined in Proposition 1 to price assets. From non-arbitrage arguments (see [15] or [16]), we know that the price of an American put option at time $t \in [0, T]$ is

$$P_t^n = \sup_{\nu \in T_{t,T}} \mathbb{E}^*[e^{-r(T-t)}(K - \tilde{S}_\nu)]_t,$$

where $T_{t,T}$ denotes the set of $\mathcal{G}$-stopping times with values in $[t, T]$. As the process $(S_t, H_t)_{t \leq T}$ is a Markov process, we have $P_t^n = P(t, S_t, H_t)$ where the value function $P$ is defined on $\mathbb{R}^+ \times \mathbb{R}^+ \times \{0, 1\}$ by

$$P(t, x, h) = \sup_{\nu \in T_{t,T}} \mathbb{E}^*[e^{-r(T-t)}f(S_\nu, H_\nu)|S_t = x; H_t = h], \quad (3)$$

where $f(x, h) = \left(K - (1-R)^h x\right)^+$ for all $(x, h) \in \mathbb{R}^+ \times \{0, 1\}$. It is well known that the process $[e^{-rt}P(t, S_t, H_t)]_{t \leq T}$ is the Snell envelope of $[e^{-rt}f(S_t, H_t)]_{t \leq T}$. Hence, we have the following result (see [17] and [18]):

**Proposition 2.** The process $[e^{-rt}P(t, S_t, H_t)]_{t \leq T}$ is the smallest RCLL super martingale which dominates $(e^{-rt}f(S_t, H_t))_{t \leq T}$ and the optimal stopping time is given by:

$$\tau^*(t, x, h) = \inf\{s \geq 0: P(s, S^{x,s}_t, H^{t,h}_s) = f(S^{x,s}_s, H^{t,h}_s)\} \wedge (T - t),$$

where $(S^{x,s}_t, H^{t,h}_s)$ is the unique solution of the equation (1) such that $S^{x,s}_t = x$ and $H^{t,h}_t = h$. Moreover, the process $[e^{-r(t \wedge \tau^*)}P(t \wedge \tau^*, S_{t \wedge \tau^*}, H_{t \wedge \tau^*})]_{t \leq T}$ is a martingale.

In the next proposition, we recall the analytical properties of the value function and its characterization as the solution of a free boundary problem.

**Proposition 3.** Analytical properties and regularity of the value function

i) The value function $P$ is non negative, for any $(t, x, h) \in [0, T] \times \mathbb{R}^+ \times \{0, 1\}$, $P(t, x, h) > 0$.

ii) For $t \in [0, T]$, the function $x \to P(t, x, h)$ is non increasing, convex and for $0 \leq x \leq y$, we have

$$0 \leq P(t, x, h) - P(t, y, h) \leq (1-R)^h(y - x).$$

iii) For $x \in [0, +\infty)$, there exists $C > 0$ such that for $0 \leq s \leq t < T$, we have

$$0 \leq P(t, x, h) - P(s, x, h) \leq C(\sqrt{T} - s - \sqrt{T - t}).$$

**Proof:** Properties i) and ii) are obvious consequences of Equation (3). Property iii) follows from the same Equation and the scaling property of the Brownian motion.
The stopping or exercise region and the continuation one need to be introduced to characterize the price of the American put option. The exercise region $E$ is defined by
\[ E = \{(t, x, h) \in [0, T) \times \mathbb{R}^+ \times \{0, 1\} | P(t, x, h) = f(x, h)\} \]
and the complement of $E$ is the continuation region $C$. We also introduce the following notations, for $h \in \{0, 1\}$,
\[ C_h = \{(t, x) \in [0, T) \times \mathbb{R}^+ | P(t, x, h) > f(x, h)\} \quad \text{and} \quad E_h = \{(t, x) \in [0, T) \times \mathbb{R}^+ | P(t, x, h) = f(x, h)\}. \]
For $(t, h) \in [0, T) \times \{0, 1\}$, we equally set $E_{t,h} = \{x \in \mathbb{R}^+ | P(t, x, h) = f(x, h)\}$. We aim at describing $E_{t,h}$ and introduce the following function:
\[ b(t, h) = \inf\{x \geq 0 : P(t, x, h) = f(x, h)\}. \]
It is now easy to deduce from Proposition 3 that
\[ E_{t,0} = [0, b(t, 0)] \quad \text{and} \quad E_{t,1} = [0, b(t, 1)]. \]
Indeed, we know that $x \to \frac{\partial P}{\partial x}(t, x, 0) \geq -1$ and that $x \to P(t, x, 1)$ is convex. From Propositions 2 and 3 we can characterize the value function $P$ as the solution of a free boundary problem.

**Proposition 4.** Let $\mathcal{L}_\lambda$ be defined as
\[ \mathcal{L}_\lambda u(t, x, h) = \frac{\partial u}{\partial t}(t, x, h) + \frac{x^2 \sigma^2(h)}{2} \frac{\partial^2 u}{\partial x^2}(t, x, h) + (r + R\gamma^*(h))x \frac{\partial u}{\partial x}(t, x, h) - ru(t, x, h). \]
The value function $P$ is a solution, in the distribution sense, of
\[ \mathcal{L}_\lambda P(t, x, h) + \gamma^*(h)[P(t, x, 1) - P(t, x, 0)] = 0 \quad \text{in } C \quad \text{(4)} \]
\[ \mathcal{L}_\lambda P(t, x, h) + \gamma^*(h)[P(t, x, 1) - P(t, x, 0)] \leq 0 \quad \text{in } [0, T) \times [0, +\infty) \times \{0, 1\} \quad \text{(5)} \]
\[ P(T, x, h) = f(x, h) \quad \text{in } [0, +\infty) \times \{0, 1\} \quad \text{(6)} \]
Moreover, $P$ is smooth in the continuation region and the smooth fit condition is satisfied. More precisely, for $h \in \{0, 1\}$, the function $(t, x) \to P(t, x, h)$ belongs to $C^{1,2}(C_h)$ and for $t \in [0, T)$, the function $x \to P(t, x, h)$ is continuously differentiable.

**Proof:** As this result is classical for American options, we just give the sketch of the proof. Equation (4) follows from the martingale property of the process $\left[e^{-r t \wedge \tau^*} P(t \wedge \tau^*, S_{t \wedge \tau^*}, H_{t \wedge \tau^*})\right]_{t \leq T}$ and the regularity of the operator $\mathcal{L}_\lambda$ (see [20] and [21]). Equation (5) is a consequence of the super martingale property of the process $\left[e^{-r t} P(t, S_t, H_t)\right]_{t \leq T}$. We deduce the smoothness of $P$ in the continuation region from Itô’s formula. Finally, the smooth fit property has been proved in [22] and in Proposition 3.2 of [20] for jump-diffusion models. It relies on the fact that the first derivatives of $P$ are locally bounded.
3.2 Optimization problem with constraint

To solve the constrained optimization problem, we follow the approach of Lacoste, Jeanblanc and El Karoui [8] and find an adapted process \((X_t)_{t \geq 0}\) such that \(V_t = X_t + \hat{P}_t \geq K\) satisfies the martingale constraint where \(\hat{P}_t\) is the price of an American put option on \(X\). This process \(X\) will be constructed thanks to the optimal wealth process of the unconstrained problem. We introduce two sets of admissible wealth processes:

\[
A(x) = \{ (V_t)_{t \geq 0} \in \mathbb{G} - \text{adapted} : V_0 = x \text{ and } (e^{-rt}V_t)_{t \geq 0} \text{ is a } \mathbb{P}^x - \text{martingale} \}
\]

and the set of admissible wealth processes with floor constraint:

\[
A_K(x) = \{ V \in A : \forall t \in [0, T], V_t \geq K \}.
\]

In the following proposition, we recall some results on the optimization problem without floor constraint (see [6], [17] and [18]).

**Proposition 5.** Let \(\nu \in \mathbb{R}\) such that \(E^x [e^{-rT}I(\nu e^{-rT}Z^*_T)] = x\) where \(I\) is the inverse function of \(U'\). We have

\[
\sup_{(V_t)_{0 \leq t \leq T} \in A(x)} E[U(V_T)] = E[U(\bar{V}_T^*)] \quad \text{where } \bar{V}_T^* = I(\nu e^{-rT}Z^*_T).
\]

The wealth \(\bar{V}_T^*\) is the unique solution of the optimization problem and is duplicable: there exists a \(\mathbb{G}\)-predictable process \((\phi_t)_{t \geq 0}\) such that

\[
e^{-rT}\bar{V}_T^* = x + \int_0^T \phi_s \frac{\bar{V}_s^*}{S_s^*} d(e^{-rS}S_s) = x + \int_0^T e^{-rS}\bar{V}_s^* \left[ \phi_s \sigma(H_s) dB_s^* - R\phi_s dN_s^* \right].
\]

Moreover, in the case of CRRA utility functions, we have from [18]:

\[
\mu(H_t) - \alpha \sigma(H_t) \phi_t = R \left[ (1 - R\phi_t)^{-\alpha} - 1 \right] (1 - H_t) \quad \forall t \in [0, T].
\]

Therefore, \(\phi_t = (1 - H_t)\phi(0) + H_t \phi(1)\) where \(\phi(1) = \mu(1)/\alpha \sigma(1)\) and \(\phi(0)\) the unique solution of the following equation,

\[
x = \frac{\mu(0)}{\alpha \sigma(0)} - \frac{R}{\alpha \sigma(0)} \left[ (1 - Rx)^{-\alpha} - 1 \right],
\]

It follows that

\[
d\bar{V}_t^* = \bar{V}_t^* (r dt + \phi(H_t) \sigma(H_t) dB_t^* - R\phi(0) dN_t^*).
\]

We define the process \(V^*_t\) as the solution of

\[
dV_t^* = V_t^* ((r + R\phi(0)\gamma^*(H_t)) dt + \phi(H_s) \sigma(H_s) dB_s^* + \phi(H_s) dN_s^*).
\]

and we introduce the value function of an American option on \(\bar{V}^*\) defined on \([0, T] \times \mathbb{R} \times \{0, 1\}\) by:

\[
\bar{P}(t, z, h) = \sup_{\nu \in \mathcal{T}_t, T} \mathbb{E}^x [e^{-r(T-t)} f(V^*_t, \ H_t) | V^*_t = z; \ H_t = h].
\]


Now, we are able to use results of Section 3.1 to construct an optimal wealth process with the floor constraint. Indeed, if we introduce the following function:

\[ V(t, z, h) = (1 - R\phi(0))^{\bar{h}} z + \bar{P}(t, z, h) \quad \text{where} \quad (t, z, h) \in [0, T] \times [0, +\infty) \times \{0, 1\}. \]

It is obvious that \( V \geq K \). Moreover, it follows from the variational inequality satisfied by \( \bar{P} \) that

\[ \bar{L}_\lambda V(t, z, h) + \gamma^*(h)(V(t, z, 1) - V(t, z, 0)) = \begin{cases} 0 & \forall (t, z, h) \in \mathcal{C} \\ -rK & \forall (t, z, h) \notin \mathcal{C}. \end{cases} \]

where we have set

\[ \bar{L}_\lambda u(t, x, h) = \frac{\partial u}{\partial t}(t, x, h) + \frac{x^2\phi(h)^2\gamma^2(h)}{2} \frac{\partial^2 u}{\partial x^2}(t, x, h) + (r + R\phi(0)\gamma^*(h))x \frac{\partial u}{\partial x}(t, x, h) - ru(t, x, h). \]

We consider the process \( Y = (e^{-rt}V(t, \pi V^*_t, H_t))_{0 \leq t \leq T} \) which corresponds to the actualised value of a long position \( \pi \) on the defaultable asset \( \bar{V}^* \) and on an American put option with strike \( K \) on the underlying \( \pi \bar{V}^* \). From Itô’s formula, it follows that the process \( Y \) is solution of the following equation:

\[
e^{rt}dY_t = \left[ \bar{L}_\lambda V(t, \pi V^*_t, H_t) + \gamma(H_t)(V(t, \pi V^*_t, 1) - V(t, \pi V^*_t, 0)) \right] dt \\
+ \pi \frac{\partial_x V(t, \pi V^*_t, H_t)}{\pi V^*_t} \left[ dV^*_t - (r + R\phi(0)\gamma(H_t))V^*_t dt + (V(t, \pi V^*_t, 1) - V(t, \pi V^*_t, 0)) dN^*_t \right] \\
= -rK \mathbb{1}_{\{\pi V^*_t \leq b(t, H_t)\}} dt + \pi \frac{\partial_x V(t, \pi V^*_t, H_t)}{\pi V^*_t} d(e^{-rt}V^*_t) + (V(t, \pi V^*_t, 1) - V(t, \pi V^*_t, 0)) dN^*_t,
\]

where we have denoted by \( b \) the free boundary associated with \( \bar{P} \). If this exercise boundary is reached before maturity by the process \( \pi \bar{V}^* \), the portfolio generates a continuous dividend \( rK \) which should be re-invested in order to have a self financing portfolio. Following [8], we will seek a process \((X_t)_{t \geq 0}\) such that \( V_t = V(t, X_t, H_t) \) is the optimal wealth under the constraint and obtain Theorem 1.

**Theorem 1.** Let \( x \geq K \) and \( a(x) \in \mathbb{R} \) be the unique solution of \( xa(x) + \bar{P}(0, xa(x), 0) = x \). For \( 0 \leq t \leq T \), we set

\[ \pi^*_t = \max \left( a(x), \sup_{0 \leq s \leq t} \frac{b(s, H_s)}{V^*_s} \right) \quad \text{and} \quad \bar{V}^*_t(K) = V(t, \pi^*_t V^*_t, H_t). \]

We have \((\bar{V}^*_t(K))_{t \geq 0} \in \mathcal{A}_K(x)\) and

\[ \max_{V \in \mathcal{A}_K(x)} \mathbb{E}[U(V_T)] = \mathbb{E}[U(\bar{V}^*_T(K))]. \]

**Proof:**

**First step:** We prove that \((\bar{V}^*_t(K))_{t \geq 0} \in \mathcal{A}_K(x)\).

We obviously have \( \bar{V}^*_t(K) \geq K \) for all \( t \in [0, T] \). Moreover, if \( x > K \) we can not have \( xa(x) \leq b(0, 0) \) because it would imply that \( x = xa(x) + \bar{P}(0, xa(x), 0) = K < x \). Therefore, we have

\[ \bar{V}^*_0(K) = \pi^*_0 x + \bar{P}(0, \pi^*_0 x, 0) = \begin{cases} K & \text{if} \quad x = K \\
xa(x) + \bar{P}(0, xa(x), H_0) & \text{if} \quad x > K. \end{cases} \]
American put option and OBPI

Notice that the process $\pi^*$ is not a continuous process since it jumps at the default time. However, it is an increasing process, growing only at time $t$ such that $\pi_t^* \bar{V}^*_t = b(t, H_t)$.

Let $X_t = \pi_t^* V_t^*$. From Ito’s calculus, we can get the dynamic of the wealth process $\bar{V}^*_t(K) = V(t, X_t, H_t)$:

$$dV(t, X_t, H_t) = \partial_t V(t, X_t, H_t) dt + \partial_x V(t, X_t, H_t) dX^c_t + \frac{1}{2} \partial_{xx} V(t, X_t, H_t) d(X^c)_t$$

$$+ (V(t, X_t, 1) - V(t, X_t -, 0)) dH_t,$$

where $X^c$ is the continuous part of the semimartingale $X$, we deduce from the integration by part formula that $dX^c_t = V^*_t d\pi^*_t + \pi^*_t dV^*_t$ then we get:

$$e^{rt} d(e^{-rt} V(t, X_t, H_t)) = (\partial_t V(t, X_t, H_t)) dt + V^*_t \partial_x V(t, X_t, H_t) d\pi^*_t$$

$$+ \pi^*_t \partial_x V(t, X_t, H_t) d\pi^*_t + V^*_t \partial_{xx} V(t, X_t, H_t) dV^*_t$$

$$+ (V(t, X_t, 1) - V(t, X_t -, 0)) (dN^*_t + \gamma(H_t) dt).$$

On the other hand, we recall that $dV^*_t = V^*_t ((r + R\phi(0)\gamma^*(H_t)) dt + \phi(H_s)\sigma(H_s) dB^*_s)$. Hence we have

$$e^{rt} d(e^{-rt} V(t, X_t, H_t)) = \tilde{L} V(t, X_t, H_t -) dt + \gamma(H_t) (V(t, X_t, 1) - V(t, X_t -, 0)) dt$$

$$+ \tilde{V}^*_t \partial_x V(t, X_t, H_t) d\pi^*_t + dB^*_t$$

where $M$ is a $\mathbb{P}^*$-martingale such that $M_0 = 0$. However, $\pi^* c$ increases only on $\{t \geq 0 : X_t = \bar{b}(t, H_t)\}$ and the smooth fit properties implies that $\partial_x V(t, b(t, H_t), H_t) = 0$. Moreover by definition of $\pi^*$ we have $X_t \geq \bar{b}(t, H_t)$ for any $t \in [0, T]$, hence $(e^{-rt} \tilde{V}^*_t(K))_{0 \leq t \leq T}$ is a $\mathbb{P}^*$-martingale and $\tilde{V}^*(K) \in \mathcal{A}_K(x)$.

**Second step**: Following the proof of [8], we show that $\tilde{V}^*(K)$ solves the optimization problem under the floor constraint. Let $V \in \mathcal{A}_K(x)$. It follows from the concavity property of $U$, that:

$$U(V_T) - U(\tilde{V}^*_T(K)) \leq U'(\tilde{V}^*_T(K))(V_T - \tilde{V}^*_T(K)).$$

Now, recall that $\tilde{V}^*_T(K) = \pi_T^* \tilde{V}^*_T + (K - \pi_T^* \tilde{V}^*_T)^+ = K \vee (\pi_T^* \tilde{V}^*_T)$. Hence we get

$$U(V_T) - U(\tilde{V}^*_T(K)) \leq U'(\pi_T^* \tilde{V}^*_T)(V_T - \tilde{V}^*_T(K)) + U'(K)(V_T - K)1_{K \leq \pi_T^* \tilde{V}^*_T} + U'(K)(V_T - K)1_{K > \pi_T^* \tilde{V}^*_T}$$

$$= U'(\pi_T^* \tilde{V}^*_T)(V_T - \tilde{V}^*_T(K)) + (U'(K) - U'(\pi_T^* \tilde{V}^*_T))(V_T - K)1_{K > \pi_T^* \tilde{V}^*_T}$$

$$\leq U'(\pi_T^* \tilde{V}^*_T)(V_T - \tilde{V}^*_T(K)).$$

The last inequality follows from the fact that $U'$ is decreasing.

As $U$ is a CRRA utility function, for any $x, y > 0$, we have $U'(xy) = U'(x)U'(y)$. Moreover, we have seen in Proposition [8] that $U'(\tilde{V}^*_T) = \nu e^{-rT} Z^*_T$, therefore we have

$$\mathbb{E}[U(V_T)] - \mathbb{E}[U(\tilde{V}^*_T(K))] \leq \nu \mathbb{E}^*[U'(\pi_T^*) e^{-rT}(V_T - \tilde{V}^*_T(K))].$$
Using integration by part formula, we get:

\[
\mathbb{E}[U(V_T)] - \mathbb{E}[U(\tilde{V}^*_T(K))] \leq \nu \mathbb{E}^* \left[ \int_0^T U'(\pi^*_s) d[e^{-rs}(V_s - \tilde{V}^*_s(K))] + \int_0^T e^{-rs}(V_s - \tilde{V}^*_s(K)) dU'(\pi^*_s) \right]
\]

Since the process \(e^{-rt}(V_t - \tilde{V}^*_t(K))\) is a \(\mathbb{P}^*\)-martingale and the measure \(dU'(\pi^*_s)\) is non positive and charges only the set \(\{\pi^*_s \tilde{V}^*_s = b(t, H_t)\}\), then we get:

\[
\mathbb{E}[U(V_T)] - \mathbb{E}[U(\tilde{V}^*_T(K))] \leq \nu \mathbb{E}^* \left[ \int_0^T e^{-rs}(V_s - K) dU'(\pi^*_s) \right] \leq 0.
\]

\[\square\]

From a practical point of view, one has now to compute the value function and the exercise boundary of an American put option. In the next section, we numerically solve this problem thanks to early exercise premium formulas.

4 Exercise boundary and early exercise premium

Throughout this section, we study the American put option on the defaultable asset introduced in Section 3.1. In order to get a numerical method to compute \(b\) and then \(P\), we will compare \(b\) to the critical price of an American put option on a non-defaultable asset. For \(h \in \{0, 1\}\), we denote by \(\hat{P}^h\) the value function of an American put option evaluated in the Black and Scholes framework on an asset with volatility \(\sigma(h)\). Let \(h \in \{0, 1\}\), we recall that:

\[
\hat{P}^h(t, x) = \sup_{\nu \in \mathcal{F}_{t,T}} \mathbb{E}^*[e^{-r(T-t)}(K - \hat{S}^h_T)_+ | \mathcal{F}_t], \quad t \leq T, \ x \in \mathbb{R},
\]

where the dynamic of the process \(\hat{S}^h\) is given by

\[
d\hat{S}^h_t = \hat{S}^h_t (r dt + \sigma(h) dB^*_t).
\]

The critical price of this option will be denoted by \(\hat{b}(t, h)\) for \((t, h) \in [0, T) \times \{0, 1\}\).

4.1 Early exercise premium, Swap premium and numerical results

4.1.1 Early exercise premium formula

We first recall the classical early exercise premium formula in the Black & Scholes model:

\[
\hat{P}^h(t, x) = \hat{P}^{e,h}(t, x) + rK \int_t^T e^{-r(s-t)} \mathbb{P}^* \left( \tilde{S}^h_s \leq \hat{b}(s, h) | \tilde{S}^h_t = x \right) ds,
\]

where \(\hat{P}^{e,h}(t, x) = \mathbb{E}^*[e^{-r(T-t)} f(\tilde{S}^h_T) | \tilde{S}^h_t = x] \) is the value function of the European put option. This formula can be used to obtain an integral equation for \(\hat{b}(., h)\), allowing us to compute \(\hat{b}(., h)\). Indeed,
for $x \leq \hat{b}(t,h)$, one has:

$$K - x = P^e_h(t,x) + rK \int_t^T e^{-r(s-t)} \mathbb{P}^* \left( \hat{S}_s^h \leq \frac{\hat{b}(s,h)}{x} | \hat{S}_t = 1 \right) ds.$$  \hspace{1cm} (7)

In this section, we will follow this idea to get an integral equation for $b(.,0)$ and then, in the next section, to derive asymptotic results for $\lambda^*$ going to 0.

Throughout this section, we will use the following notation:

$$\mathbb{E}_{t,h,x}^* [ \cdot ] = \mathbb{E} [ \cdot | S_t = x, H_t = h ] \quad \forall (t,x,h) \in [0,T] \times \mathbb{R}^+ \times \{0,1\}.$$

**Proposition 6.** In our framework, the exercise premium formula is given by:

$$P(t,x,0) = P^e(t,x,0) + rK \int_t^T e^{-r(s-t)} \mathbb{P}_{t,x,0}^*(S_s \leq b(s,H_s)) ds$$

$$- \lambda^* \mathbb{E}_{t,x,0}^* \left[ \int_t^T e^{-r(s-t)} (P(s,S_s,1) - f(S_s,1)) 1_{\{ S_s \leq b(s,0) \}} 1_{\{ H_s = 0 \}} ds \right],$$

where $P^e$ is the value function of the European put option and is defined by

$$P^e(t,x,h) = \mathbb{E}_{t,x,h}[e^{-r(T-t)} f(S_T, H_T)].$$

**Proof:** Let $t \in [0,T)$. As $P$ is continuously differentiable with respect to time and space and is convex with respect to the space variable, we can apply Itô’s formula to the process $(e^{-r_s}P(s,S_s,H_s))_{0 \leq s \leq T}$ between $t$ and $T$. We get:

$$e^{-rT} f(S_T, H_T) = e^{-rt} P(t,x,0) + M_T - M_t$$

$$+ \int_t^T e^{-rs} (\mathcal{L}_\lambda P(s,S_s,H_s) + \gamma^*(H_s) (P(s,S_s,1) - P(s,S_s,0))) ds.$$

where the process $M$ is the martingale defined by

$$M_t = \int_0^t e^{-rs} \partial_x P(s,S_s,H_s) \sigma(S_s) dB_s^x + \int_0^t e^{-rs} (P(s,S_s,1) - P(s,S_s,0)) dN_s^x, \quad t \leq T.$$

From Proposition 4, we know that for $(s,h) \in [t,T] \times \{0,1\}$ and $x \in (b(s,h), +\infty)$, we have

$$\mathcal{L}_\lambda P(s,x,h) + \gamma^*(h) (P(s,x,1) - P(s,x,0)) = -rK - \gamma^*(h) (RS_s - P(s,S_s,1) + K - S_s)$$

$$= -rK + \gamma^*(h) (P(s,S_s,1) - f(S_s,1)).$$

As we have $P(s,x,1) = f(x,1)$ for $x \leq b(s,1)$, by taking the conditional expectation with respect to $\{ S_t = x, H_t = 0 \}$ in the first equation of the proof, we finally obtain

$$P^e(t,x,0) = P(t,x,0) - rK \mathbb{E}_{t,x,0}^* \left[ \int_t^T e^{-r(s-t)} 1_{\{ S_s \leq b(s,H_s) \}} ds \right]$$

$$+ \lambda^* \mathbb{E}_{t,x,0}^* \left[ \int_t^T e^{-r(s-t)} (P(s,S_s,1) - f(S_s,1)) 1_{\{ S_s \leq b(s,0) \}} 1_{\{ H_s = 0 \}} ds \right].$$
4.1.2 American swap premium

In this section we derive a formula for the default premium, the difference between the price of a European put option on a defaultable asset and the one a free-default asset. From this formula, we derive an integral equation satisfied by \( b \) and use it to give numerical approximations of \( b \) and \( P \).

**Proposition 7.** Let \((t,x) \in [0,T) \times [0, +\infty)\). We have the following equation

\[
P^a(t,x,0) = \hat{\mathcal{P}}^e(t,x) + e(t,x) + d(t,x),
\]

where the early exercise premium is

\[
e(t,x) := rK \int_t^T e^{-r(s-t)} \mathbb{E}^*_{t,x,0} (S_s \leq b(s,H_s)) \, ds
\]

\[
-\lambda^* \mathbb{E}^*_{t,x,0} \left[ \int_t^T e^{-r(s-t)} (P(s,S_s,1) - f(S_s,1)) 1_{\{S_s \leq b(s,0)\}} 1_{\{H_s=0\}} \, ds \right],
\]

and if we set \( \Delta \hat{\mathcal{P}}^e(t,x) = \hat{\mathcal{P}}^e(t,x(1-R)) - \hat{\mathcal{P}}^e(t,x) + R x \partial_x \hat{\mathcal{P}}^e(t,x) \), the default premium is

\[
d(t,x) := \mathbb{E}^*_{t,x,0} \left[ \int_t^T e^{-r(s-t)} \frac{(\sigma(1)^2 - \sigma(0)^2)S^2_s}{2} \partial^2 \hat{\mathcal{P}}^e(s,S_s,1) \, ds \right]
\]

\[
+ \lambda^* \int_t^T e^{-(r+\lambda^*)(s-t)} \mathbb{E}^* \left[ \Delta \hat{\mathcal{P}}^e(s,e^\lambda^*S_0^0) 1_{H_s=1} \right] \, ds.
\]

**Proof:** Let \( t \in [0,T) \). As \( \hat{\mathcal{P}}^e \) is a \( C^{1,2} \) function on \([0,T) \times [0, +\infty)\) we can apply Itô formula to the process \((e^{-rs} \hat{\mathcal{P}}^e(s,(1-R)H_s,S_s))_{0 \leq s \leq T}\) between \( t \) and \( T \) and take the conditional expectation with respect to \( \{S_t = x, H_t = 0\} \). We obtain

\[
P^e(t,x,0) = \hat{\mathcal{P}}^e(t,x) + \mathbb{E}^*_{t,x,0} \left[ \int_t^T e^{-r(s-t)} \mathcal{L} \hat{\mathcal{P}}^e(s,S_s) \, ds \right]
\]

\[
+ \mathbb{E}^*_{t,x,0} \left[ \int_t^T e^{-r(s-t)} \gamma^*(H_s) \left( \hat{\mathcal{P}}^e(s,S_s(1-R)) - \hat{\mathcal{P}}^e(s,S_s) \right) \, ds \right]
\]

\[
= \hat{\mathcal{P}}^e(t,x) + \mathbb{E}^*_{t,x,0} \left[ \int_t^T e^{-r(s-t)} \frac{(\sigma(1)^2 - \sigma(0)^2)S^2_s}{2} \partial^2 \hat{\mathcal{P}}^e(s,S_s,1) \, ds \right]
\]

\[
+ \lambda^* \mathbb{E}^*_{t,x,0} \left[ \int_t^T e^{-r(s-t)} 1_{H_s=0} \Delta \hat{\mathcal{P}}^e(s,S_s) \, ds \right].
\]

Moreover, on \( \{H_s = 0\} \), we have \( S_s = e^{R \lambda^* S_0^0} \) then we get

\[
\mathbb{E}^*_{t,x,0} \left[ \int_t^T e^{-r(s-t)} 1_{H_s=0} \Delta \hat{\mathcal{P}}^e(s,S_s) \, ds \right] = \int_t^T e^{-(r+\lambda^*)(s-t)} \mathbb{E}^* \left[ \Delta \hat{\mathcal{P}}^e(s,e^{R \lambda^* S_0^0}) \, ds \right] \left| e^{\lambda^* S_0^0} \right| = x.
\]

Finally, the result follows from Proposition 8.
Remark 1. The premium $e$ and $d$ are not positive functions. Their signs strongly depend on the jump of the volatility $\sigma(0) - \sigma(1)$ compared to the asset one. We can notice that in the case $\sigma(0) = \sigma(1)$, we have

$$e(t, x) = rK \int_t^T e^{-r(s-t)} \mathbb{P}^*_{t,x,0} (S_s \leq b(s, H_s)) \, ds$$

$$d(t, x) = \lambda^* \int_t^T e^{-(r+\lambda^*)(s-t)} \mathbb{E}[\Delta \hat{P}^e,0(s, e^{\lambda^*}s \hat{S}_s^0)| e^{\lambda^*t} \hat{S}_0^0 = x] \, ds.$$

Indeed, we will see in Proposition 8 that when $\sigma(0) \leq \sigma(1)$, $b(., 0) \leq \hat{b}(., 0) = \hat{b}(., 1) = (1 - R)b(.,)$. Hence, on $\{S_s \leq b(s, 0)\}$, we have $P(s, S_s, 1) = f(S_s, 1)$.

From Proposition 7 we can derive the following integral equation satisfied by $b(t, 0)$:

$$K - b(t, 0) = \hat{P}^e(t, b(t, 0)) + e(t, b(t, 0)) + d(t, b(t, 0)).$$

We give in the next figures some numerical results. First we compute the free boundaries with and without default thanks to the integral equations, then using the early exercise premium formula we get the prices. We consider the following parameter: maturity of the option $T = 1$, the volatility $\sigma(1) = \sigma(0) = 0.2$, the interest rate $r = 0.02$, the recovery rate $R = 0.4$. In figure 1, we plot the free boundaries functions with and without the default: $t \rightarrow \hat{b}(t)$ and $t \rightarrow b(t, 0)$.

![Free Boundary with Default](image1.png)

**Figure 1: Free Boundaries**

In figure 2, since we numerically find the free boundaries, we deduce from the early exercise price formula the price of an American put option on asset with and without default.

However, the integral equation satisfied by $b(., 0)$ does not lead to a simple computation method. We give in the next section asymptotic estimations for $\lambda^*$ going to 0. As the swap premium goes to zero when the intensity goes to zero, it implies that the American put option price converges to the price of a classical American put option. Moreover, we give an estimation of the convergence rates for the price and boundaries when $\lambda^*$ converges to zero.
4.2 Asymptotic behavior of the exercise boundary

In this section we want to study the sensitivity to the american put option on a defaultable asset with respect to the default intensity. We will prove that when the intensity goes to zero then the put option is a classical put written on a free-default asset plus an error that we can estimate. In the next Proposition, we give estimations of the functions $P - \hat{P}$ and $b - \hat{b}$.

**Proposition 8.** For $(t,x) \in [0, T) \times [0, +\infty)$, we have

$$P(t, x, 1) = \hat{P}(t, (1 - R)x) \text{ and } (1 - R)b(t, 1) = \hat{b}(t, 1)$$

Moreover, we have the following results.

- If $\sigma(0) < \sigma(1)$, there exist $a > 0$ and $b > 0$ such that

  $$0 \leq P(t, x, 0) - \hat{P}^0(t, x) \leq \lambda^*(ax + b)\sqrt{T - t} \quad \text{for all } (t, x) \in [0, T) \times [0, +\infty).$$

  There exists $C > 0$, such that for $\lambda^*$ going to 0, we have

  $$0 \leq \hat{b}(t, 0) - b(t, 0) \leq C\sqrt{\lambda^*} + o(\sqrt{\lambda^*}) \quad \text{for all } t \in [0, T).$$

- If $\sigma(1) \leq \sigma(0)$, there exist $c > 0$ and $d > 0$ such that

  $$-\lambda^*(cx + d)\sqrt{T - t} \leq P(t, x, 0) - \hat{P}^0(t, xe^{R\lambda^*(T-t)}) \leq x(e^{R\lambda^*(T-t)} - 1) \quad \text{for all } (t, x) \in [0, T) \times [0, +\infty).$$

  There exists $C > 0$, such that for $\lambda^*$ going to 0, we have

  $$0 \leq b(t, 0) - \hat{b}(t, 0)e^{-R\lambda^*(T-t)} \leq C\sqrt{\lambda^*} + o(\sqrt{\lambda^*}) \quad \text{for all } t \in [0, T).$$
Proof: see the Appendix.

**Remark 2.** If the volatility does not jump at the default time ($\sigma(0) = \sigma(1)$), we have the following inequality:

$$\hat{b}(t, 0)e^{-R\lambda^*(T-t)} \leq b(t, 0) \leq \hat{b}(t, 0).$$

We now give some graphics to show the asymptotic results when the default intensity goes to zero. In figure 3 and figure 4, we draw the free boundaries and the put option prices with respect to the default intensity, for a fixed time $t \in [0, T]$. In figure 4, we draw the prices of the American put option on a defaultable asset with respect to the intensity value. We find numerically a linear approximation of the price of the put option with respect to the value intensity.

![Free Boundary convergence](image-url)

**Figure 3:** Free Boundaries convergence
Figure 4: American put options prices convergence

5 APPENDIX

5.1 Proof of Proposition. [1]

The dual function of a CRRA utility function with a parameter denoted $\alpha \in (0,1)$ is given by:

$$\tilde{U}(y) = \max_{x > 0} \{ U(x) - xy \} = U[I(y)] - yI(y) = \kappa y^{-\kappa}, \quad y > 0$$

where $\kappa = \frac{1-\alpha}{\alpha} > 0$. Hence to get the optimal risk neutral density, we should find a process $\delta > -1$ minimizing $\mathbb{E} \left[ \nu \kappa (Z_T^\delta)^{-\kappa} \right]$ where $Z^\delta$ is solution of

$$\begin{cases}
  dZ_t^\delta = Z_t^\delta \left[ \beta(\delta_t, H_t) dB_t + \delta_t dN_t \right], \\
  Z_0^\delta = 1,
\end{cases}$$

with $\beta(\delta_t, H_t) = \frac{1}{\sigma(H_t)}(r - \mu(H_t) + R\delta_t \gamma(H_t))$. (8)

Let $\delta > -1$ be a stochastic process such that the equation (8) admits a solution. It is important to notice that on $\{ \tau < t \}$, we have $\beta(1) := \beta(\delta_t, 1) = (r - \mu(1))/\sigma(1)$ and $\delta_t = 0$.

From Itô’s formula, we get:

$$\begin{align*}
\mathbb{E}(Z_T^\delta)^{-\kappa} &= 1 + \mathbb{E} \int_0^T (Z_t^\delta)^{-\kappa} \left[ \frac{\kappa(\kappa+1)}{2} \beta^2(\delta_t, H_t) + ((1 + \delta_t)^{-\kappa} - 1 + \kappa \delta_t) \lambda(1 - H_t) \right] dt \\
&= 1 + \mathbb{E} \int_0^T (Z_t^\delta)^{-\kappa} \left[ F(\delta_t) \mathbb{I}_{\{H_t=0\}} + \frac{\kappa(\kappa+1)}{2} \beta^2(1) \mathbb{I}_{\{H_t=1\}} \right] dt
\end{align*}$$

where we have set

$$F(x) = \frac{\kappa(\kappa+1)}{2} \beta^2(x, 0) + ((1 + x)^{-\kappa} - 1 + \kappa x) \lambda, \quad \text{on } \mathbb{R}.$$
We define \( x^* \) such that
\[
\forall x \in \mathbb{R}, \quad F(x^*) \leq F(x).
\]
It is easy to check that \( x^* \) is the unique solution of the following equation on \((-1, +\infty)\).
\[
(\kappa + 1) \lambda \frac{R^2}{\sigma(0)^2} x - (x + 1)^{-(\kappa + 1)} = (\kappa + 1) \frac{R}{\sigma(0)^2} (\mu(0) - r) - 1.
\]
We define \( Z^* \) as the solution of equation \(8\) associated to the coefficient \( \delta^*_t := x^* \mathbb{I}_{\{H_t = 0\}} \) and denote by \( \Delta^\delta \) the process defined by \( \Delta^\delta_t = (Z^*_t)^{1-\kappa} - (Z^*_0)^{1-\kappa} \) on \([0, T]\). It follows from the Itô formula and the previous computations that
\[
\mathbb{E}[\Delta^\delta_T | \mathcal{G}_t] \geq \int_0^T \mathbb{E} \left[ \Delta^\delta_t F(\delta^*_t) | \mathcal{G}_t \right] \mathbb{I}_{\{H_t = 0\}} dt + \frac{\kappa(\kappa + 1)}{2} \beta^2(1) \int_0^T \mathbb{E}[\Delta^\delta_t | \mathcal{G}_t] \mathbb{I}_{\{H_t = 1\}} dt,
\]
On the other hand, we have
\[
F(x^*) \geq (1 + x^*)^{-\kappa} - 1 + \kappa x^* \lambda \geq 0,
\]
Therefore, we can apply Gronwall lemma to assert that \( \mathbb{E}[(Z^*_T)^{1-\kappa} - (Z^*_0)^{1-\kappa} | \mathcal{G}_T] \geq 0 \). We conclude the proof by taking the expectation in this last inequality.

### 5.2 Proof of Proposition 8

It is easy to deduce from equation \(3\) \( P(t, x, 1) = \hat{P}^1(t, (1 - R)x) \) for all \((t, x) \in [0, T] \times [0, +\infty)\). It implies that \( (1 - R)\hat{b}(., 1) = \hat{b}(., 1) \).

**First case:** We assume that \( \sigma(0) \leq \sigma(1) \).

**First step:** We prove that \( \hat{P}^0(t, x) \leq P(t, x, 0) \) for all \((t, x) \in [0, T] \times [0, +\infty)\).

We follow ideas of the proof of Proposition 5.1 in \([20]\) and set \( F(t, x, h) = P(t, x, h) - \hat{P}^h(t, (1 - R)^h x) \). Notice that we have proved that \( F(t, x, 1) = 0 \).

If \( x \leq \hat{b}(t, 0) \), we have \( F(t, x, 0) = P(t, x, 0) - f(x, 0) \geq 0 \). Assume that \( x > \hat{b}(t, 0) \). We have \( \mathcal{L}_x P(t, x, 0) \leq -\lambda^* (P(t, x, 1) - P(t, x, 0)) \) then we get:
\[
\mathcal{L}_x F(t, x, 0) + \lambda^*[F(t, x, 1) - F(t, x, 0)] \leq \lambda^* \left( \hat{P}^0(t, x) - \hat{P}^1(t, (1 - R)x) - R x \frac{\partial \hat{P}^0}{\partial x}(t, x) \right)
\]
\[
\leq \lambda^* \left( \hat{P}^0(t, x) - \hat{P}^0(t, (1 - R)x) - R x \frac{\partial \hat{P}^0}{\partial x}(t, x) \right)
\]
Indeed, we have \( \hat{P}^1 \geq \hat{P}^0 \) since \( \sigma(0) \leq \sigma(1) \). It follows from the convexity of \( \hat{P}^0(t, .) \) that
\[
\mathcal{L}_x F(t, x) + \lambda^*[F(t, x, 1) - F(t, x, 0)] \leq 0
\]
and from the maximum principle, we obtain that $F(t, x, 0) \geq 0$.
Consequently, we obviously have $b(t, 0) \leq \hat{b}(t, 0)$.

**Second step:** We prove that there exist $a > 0$ and $b > 0$ such that

$$P(t, x, 0) - \hat{P}^0(t, x) \leq \lambda^*(ax + b) \quad \forall (t, x) \in [0, T) \times [0, +\infty).$$

Let $(t, x) \in [0, T) \times [0, +\infty)$. As $P(., ., 1) \geq f(., 1)$, it follows from Proposition 7 that

$$P(t, x, 0) - \hat{P}^0(t, x) \leq I_1(t, x) + I_2(t, x) + I_3(t, x),$$

where we have set

$$I_1(t, x) = rK \int_t^T e^{-r(s-t)} \left( \mathbb{P}_{s,t,x}^* (S_s \leq b(s, H_s)) - \mathbb{P}_x^* (\hat{S}_t^0 \leq \hat{b}(s, 0) | \hat{S}_t^0 = x) \right) ds$$

$$I_2(t, x) = \mathbb{E}_{t,x,0}^* \left[ \int_t^T e^{-r(s-t)} \left( \frac{(\sigma(1))^2 - \sigma(0)^2}{2} \frac{\partial^2 \hat{P}^0}{\partial x^2} (s, S_s) \mathbb{I}_{\{H_s = 1\}} \right) ds \right]$$

$$I_3(t, x) = \lambda^* \int_t^T e^{-(r + \lambda^*)(s-t)} \mathbb{E}^*[\Delta \hat{P}^e, 0 (s, e^{\lambda^* S_t^0}) | e^{\lambda^* t} \hat{S}_t^0 = x] ds.$$

Now, we have to find upper bounds for $I_i$. As $b(t, 0) \leq \hat{b}(t, 0)$, we have

$$I_1(t, x) = rK \int_t^T e^{-r(s-t)} \mathbb{E}^*[\left( \mathbb{I}_{\{s \in \mathbb{R}^+ \}, S_s \leq b(s, 0)} - \mathbb{I}_{\{S_s \leq \hat{b}(s, 0)\}} \right) \mathbb{I}_{\{H_s = 0\}} | \hat{S}_t^0 = x, H_t = 0] ds$$

$$+ rK \int_t^T e^{-r(s-t)} \mathbb{E}^*[\left( \mathbb{I}_{\{S_s \leq \hat{b}(s, 0)\}} - \mathbb{I}_{\{S_s \leq b(s, 0)\}} \right) \mathbb{I}_{\{H_s = 1\}} | \hat{S}_t^0 = x, H_t = 0] ds.$$

$$\leq - rK \int_t^T e^{-r(s-t)} \mathbb{E}^*[\left( e^{-R^* \lambda^* (s-t)b(s, 0)} \leq \hat{b}(s, 0); \ H_s = 0 | \hat{S}_t^0 = x, H_t = 0 \right) ds$$

$$+ rK \int_t^T e^{-r(s-t)} \mathbb{E}^*[H_s = 1 | H_t = 0] ds.$$

Therefore, we have $I_1(t, x) \leq \lambda^* K(T - t)$. On the other hand, for $s \in [t, T)$, we have

$$S_s^2 \frac{\partial^2 \hat{P}^e, 0}{\partial x^2} (s, S_s) \leq \frac{S_s}{\sigma(0) \sqrt{2\pi(s-t)}}.$$

Hence, as $\sigma(0) \leq \sigma(1)$, we get

$$I_2(t, x) \leq \int_t^T \frac{(\sigma(1))^2 - \sigma(0)^2}{2\sigma(0) \sqrt{2\pi(s-t)}} \mathbb{E}_{t,x,0}^* \left[ e^{-r(s-t)} S_s \mathbb{I}_{\{H_s = 1\}} \right] ds.$$

Chevalier and Ngoupeyou
The processes \((e^{-rs}(1 - R)^H S_s)_{0 \leq s \leq T}\) and \((e^{-rs} \tilde{S}^0_s)_{0 \leq s \leq T}\) \(\mathbb{P}^*\)-martingales, so it follows that

\[
(1 - R)E^*_{t,x,0} \left[ e^{-r(s-t)} S_s \mathbb{I}_{\{H_s = 1\}} \right] = E^*_{t,x,0} \left[ e^{-r(s-t)}(1 - R)^H S_s \mathbb{I}_{\{H_s = 1\}} \right] = \left( x - E^*_{t,x,0} \left[ e^{-r(s-t)} S_s \mathbb{I}_{\{H_s = 0\}} \right] \right) = \left( x - E^* \left[ e^{-r(s-t)} e^{R \lambda^* (s-t)} \tilde{S}_s^0 \mathbb{I}_{\{H_s = 0\}} | \tilde{S}_t^0 = x e^{-R \lambda^* t}; \ H_t = 0 \right] \right) = \left( x - E^* \left[ e^{-r(s-t)} e^{R \lambda^* (s-t)} \tilde{S}_s^0 \mathbb{I}_{\{H_s = 0\}} | \tilde{S}_t^0 = x e^{-R \lambda^* t}; \ H_t = 0 \right] \right) = x \left( 1 - e^{-R \lambda^* t} \right)
\]

We obtain that

\[
I_2(t, x) \leq \frac{x R \lambda^* (\sigma(1)^2 - \sigma(0)^2) \sqrt{T - t}}{\sigma(0)(1 - R) \sqrt{2\pi}}
\]

Finally, from the Black and Scholes formula, we deduce that

\[
\Delta \hat{P}^{e,0}(s, y) = e^{-r(T-s)} K \left( \mathcal{N} \left( d_1 - \frac{\ln(1 - R)}{\sigma(0) \sqrt{T - s}} \right) - \mathcal{N}(d_1) \right) - y(1-R) \left( \mathcal{N} \left( d_2 - \frac{\ln(1 - R)}{\sigma(0) \sqrt{T - s}} \right) - \mathcal{N}(d_2) \right),
\]

where \(d_1 = \left( \ln(K/y) - (r - \frac{\sigma(0)^2}{2})(T-s) \right) / \sigma(0) \sqrt{T-s}\) and \(d_2 = d_1 - \sigma(0) \sqrt{T-s}\). Hence, we have

\[
\Delta \hat{P}^{e,0}(s, y) \leq \frac{K |\ln(1 - R)|}{\sigma(0) \sqrt{2\pi(T-s)}}
\]

We obtain that

\[
I_3(t, x) \leq \lambda^* \frac{2K |\ln(1 - R)| \sqrt{T-t}}{\sigma(0) \sqrt{2\pi}}
\]

In conclusion, we have proved that there exist two constants \(a > 0\) and \(b > 0\) which do not depend on \(t\) and \(x\) such that

\[
P(t, x, 0) - \hat{P}^0(t, x) \leq \lambda^* (ax + b) \sqrt{T-t}.
\]

**Third step:** We prove that there exists \(C > 0\) which does not depend on \(t\) such that

\[
0 \leq \hat{b}(t, 0) - b(t, 0) \leq C \sqrt{x^*} + o(\sqrt{x^*}).
\]

Let \(t \in [0, T)\) and \(x \in (b(t, 0), \hat{b}(t, 0))\). From the variational inequality satisfied by \(P\), we know that:

\[
\frac{\partial P}{\partial t}(t, x, 0) + \frac{x^2 \sigma^2(0)}{2} \frac{\partial^2 P}{\partial x^2}(t, x, 0) + (r + R \lambda^*) x \frac{\partial P}{\partial x}(t, x, 0) - r P(t, x, 0) + \lambda^* (P(t, x, 1) - P(t, x, 0)) = 0.
\]
As \( s \to P(s, x, 0) \) is non-increasing, \( y \to P(t, y, 0) \) is non-increasing and convex, we have
\[
\frac{K^2 \sigma^2(0)}{2} \frac{\partial^2 P}{\partial x^2}(t, x, 0) \geq \frac{x^2 \sigma^2(0)}{2} \frac{\partial^2 P}{\partial x^2}(t, x, 0)
\]
\[
\geq -rx \frac{\partial P}{\partial x}(t, x, 0) + rP(t, x, 0) - \lambda^*(P(t, x, 1) - P(t, x, 0)) + Rx \frac{\partial P}{\partial x}(t, x, 0)
\]
\[
\geq -rx \frac{\partial P}{\partial x}(t, x, 0) + r(K - x) - \lambda^*(K - (K - x))
\]
\[
\geq \left( rK - \lambda^* \hat{b}(t, 0) \right) - rx \left( \frac{\partial P}{\partial x}(t, x, 0) + 1 \right)
\]
\[
\geq \left( rK - \lambda^* \hat{b}(t, 0) \right) - rK \left( \frac{\partial P}{\partial x}(t, x, 0) + 1 \right).
\]

Now, we integrate the last inequality between \( b(t, 0) \) and \( y \in (b(t, 0), \hat{b}(t, 0)) \) and use the smooth fit property to obtain
\[
\frac{K^2 \sigma^2(0)}{2} \left( \frac{\partial P}{\partial x}(t, x, 0) + 1 \right) \geq \left( rK - \lambda^* \hat{b}(t, 0) \right)(y - b(t, 0)) - rK (P(t, y, 0) - f(y, 0)).
\]

It follows from the previous step that \( P(t, y, 0) - f(y, 0) = P(t, y, 0) - \hat{P}^0(t, y) \leq \lambda^*(ay + b) \), then we have
\[
\frac{K^2 \sigma^2(0)}{2} \left( \frac{\partial P}{\partial x}(t, x, 0) + 1 \right) \geq \left( rK - \lambda^* \hat{b}(t, 0) \right)(y - b(t, 0)) - rK \lambda^*(aK + b).
\]

Once again, we integrate this inequality between \( b(t, 0) \) and \( \hat{b}(t, 0) \) and get
\[
\frac{K^2 \sigma^2(0)}{2} \left( P(t, \hat{b}(t, 0), 0) - f(\hat{b}(t, 0), 0) \right) \geq \frac{1}{2} \left( rK - \lambda^* \hat{b}(t, 0) \right)(\Delta b(t))^2 - rK \lambda^*(aK + b)(\Delta b(t)).
\]

where we have set \( \Delta b(t) = \hat{b}(t, 0) - b(t, 0) \). From the previous step, we deduce that
\[
\frac{K^2 \sigma^2(0)}{2} \lambda^*(aK + b) \geq \frac{1}{2} \left( rK - \lambda^* \hat{b}(t, 0) \right)(\Delta b(t))^2 - rK \lambda^*(aK + b)(\Delta b(t)).
\]

It implies that for \( \lambda^* \) going to 0, there exists \( C > 0 \) which does not depend on \( t \) such that
\[
0 \leq \hat{b}(t, 0) - b(t, 0) \leq C\sqrt{\lambda^*} + o(\sqrt{\lambda^*}).
\]

Second case: We assume that \( \sigma(0) > \sigma(1) \).

First step: We prove that \( P(t, x, 0) \leq \hat{P}^0(t, xe^{R\lambda^*(T-t)}) + x(e^{R\lambda^*(T-t)} - 1) \) for all \( (t, x) \in [0, T) \times [0, +\infty) \).

We set \( G(t, x, h) = \hat{P}^h(t, x(1-R)^h e^{R\lambda^*(T-t)}) + x(1-R)^h(e^{R\lambda^*(T-t)} - 1) - P(t, x, h) \). We have
\[
G(t, x, 1) = \hat{P}^0(t, x(1-R)e^{R\lambda^*(T-t)}) + x(1-R)(e^{R\lambda^*(T-t)} - 1) - P(t, x, 1)
\]
\[
= \hat{P}^0(t, x(1-R)e^{R\lambda^*(T-t)}) + x(1-R)(e^{R\lambda^*(T-t)} - 1) - \hat{P}^1(t, (1-R)x)
\]
\[
\geq \hat{P}^1(t, x(1-R)e^{R\lambda^*(T-t)}) + x(1-R)(e^{R\lambda^*(T-t)} - 1) - \hat{P}^1(t, (1-R)x).
\]
From the convexity of $\hat{P}^1(t,.)$, it is easy to see that $G(t,x,1) \geq 0$. Moreover, $G(T,x,0) \geq 0$ and for $x \leq b(t,0)$, we have $G(t,x,0) = \hat{P}^0(t,x) - R_x e^{-R_x T} = 0$.

Assume that $x > b(t,0)$. We have

$$AG(t,x,0) := \mathcal{L}_x G(t,x,0) + \lambda^*(G(t,x,1) - G(t,x,0))$$

$$= \mathcal{L}_x \left[ \hat{P}^0(t,x) + x (e^{R_x T} - 1) \right]$$

$$+ \lambda^* \left( \hat{P}^1(t,x(1-R)) - \hat{P}^0(t,x) - R_x e^{-R_x T} \right)$$

$$= \mathcal{L}_x \hat{P}^0(t,x) + x R_x - \lambda^* \left( \hat{P}^1(t,x(1-R)) - \hat{P}^0(t,x) - R_x e^{-R_x T} \right)$$

$$\leq \lambda^* \left( \hat{P}^1(t,x(1-R)) - \hat{P}^0(t,x) - R_x e^{-R_x T} \right)$$

$$\leq \lambda^* \left( \hat{P}^0(t,x(1-R)) - \hat{P}^0(t,x) - R_x e^{-R_x T} \right).$$

The convexity of $\hat{P}^0$ implies that, for $y \in [0, +\infty)$,

$$\hat{P}^0(t,y) \geq \hat{P}^0(t, (1-R)y) + R_y \frac{\partial \hat{P}^0}{\partial x}(t, (1-R)y) \geq \hat{P}^0(t, (1-R)y) - R_y.$$

Hence, we have $\mathcal{L}_x G(t,x,0) + \lambda^*(G(t,x,1) - G(t,x,0)) \leq 0$ and from the maximum principle, we get $G(t,x,0) \geq 0$.

Now, we have

$$P(t,b(t,0))e^{-R_x T} \leq \hat{P}(t,\hat{b}(t,0)) + \hat{b}(t,0)(1 - e^{-R_x T}) = K - \hat{b}(t,0)e^{-R_x T},$$

then $\hat{b}(t,0)e^{-R_x T} \leq b(t,0)$.

**Second step:** We prove that there exist $a > 0$ and $b > 0$ such that

$$\hat{P}^0(t,x) - P(t,x,0) \leq \lambda^*(ax + b) \quad \forall (t,x) \in [0,T] \times [0, +\infty).$$

Let $(t,x) \in [0,T] \times [0, +\infty)$. From the convexity of $\hat{P}^e,0$, we know that $\Delta \hat{P}^e,0 \geq 0$, it follows from Proposition 7 that

$$\hat{P}^0(t,x) - P(t,x,0) \leq J_1(t,x) + J_2(t,x) + J_3(t,x) + J_4(t,x),$$

where we have set

$$J_1(t,x) = \hat{P}^e,0(t,x) - \hat{P}^e,0(t,x),$$

$$J_2(t,x) = rK \int_t^T e^{-r(s-t)} \left( \mathbb{P}^x \left( S^0_s = \hat{b}(s,0) \right) \hat{S}^0_t = x e^{R_x (T-t)} \right) - \mathbb{P}^x_{t,x,0} \left( S_s \leq b(s,H_s) \right) ds,$$

$$J_3(t,x) = \lambda^x \mathbb{E}^x_{t,x,0} \left[ \int_t^T e^{-r(s-t)} \left( P(s,S_s,1) - f(S_s,1) \right) \mathbb{1}_{\{S_s \leq b(s,H_s)\}} \mathbb{1}_{\{H_s = 0\}} ds \right],$$

$$J_4(t,x) = \mathbb{E}^x_{t,x,0} \left[ \int_t^T e^{-r(s-t)} \left( \sigma(0)^2 - \sigma(1)^2 \right) S_s^2 \frac{\partial^2 \hat{P}^e,0}{\partial x^2} (s,S_s) \mathbb{1}_{\{H_s = 1\}} ds \right].$$
Now, we have to find upper bounds for $J_1$. As $\hat{P}^{e,0}$ is non-increasing, we obviously have $J_1 \leq 0$. As for $s \in [0, T)$, $b(s, 0) e^{-R\lambda^*(T-s)} \leq b(s, 0)$, we have

$$J_2(t, x) = rK \int_t^T e^{-r(s-t)} E^s \left[ \mathbb{I}_{\{S_t^0 \leq b(s,0) e^{-R\lambda^*(T-t)}\}} - \mathbb{I}_{\{S_t^0 \leq b(s,0)\}} \right] \mathbb{I}_{\{H_s=0\}} |\hat{S}_t^0 = x, H_t = 0| ds$$

$$+ rK \int_t^T e^{-r(s-t)} E^s \left[ \mathbb{I}_{\{S_t^0 \leq b(s,0)\}} - \mathbb{I}_{\{S_s \leq b(s,1)\}} \right] \mathbb{I}_{\{H_s=1\}} |\hat{S}_t^0 = x, H_t = 0| ds.$$  

$$\leq -rK \int_t^T e^{-r(s-t)} P^s \left( \hat{b}(s, 0) e^{-R\lambda^*(T-s)} \leq \hat{S}_s^0 \leq \hat{b}(s, 0); H_s = 0| \hat{S}_s^0 = x, H_t = 0 \right) ds$$

$$+ rK \int_t^T e^{-r(s-t)} (1 - e^{-\lambda(s-t)}) ds.$$  

Therefore, we have $J_2(t, x) \leq \lambda^* K(T-t)$. On the other hand, $P \leq K$ and $b \leq K$, then, for $s \in [t, T)$, we have

$$(P(s, s_s, 1) - f(s, 1)) \mathbb{I}_{\{S_s \leq b(s,0)\}} \leq (1 - R)K.$$  

Hence, we get $J_3 \leq (1 - R)K\lambda^*(T-t)$.

Finally, we have seen in the second step of the first case that

$$J_4 \leq \frac{xR\lambda^*(\sigma(0)^2 - \sigma(1)^2)\sqrt{T-t}}{\sigma(0)(1-R)\sqrt{2\pi}}.$$  

In conclusion, we have proved that there exist two constants $c > 0$ and $d > 0$ which do not depend on $t$ and $x$ and such that

$$\hat{P}^0(t, xe^{R\lambda^*(T-t)}) - P(t, x, 0) \leq \lambda^*(cx + d)\sqrt{T-t}.$$  

**Third step:** We prove that there exists $C > 0$ which does not depend on $t$ such that

$$0 \leq b(t, 0) - \hat{b}(t, 0) e^{-R\lambda^*(T-t)} \leq C\sqrt{x} + o(\sqrt{x}).$$  

Let $t \in [0, T)$ and $x \in (\hat{b}(t, 0) e^{-R\lambda^*(T-t)}, b(t, 0))$. From the variational inequality satisfied by $\hat{P}^0$, we know that:

$$\frac{\partial \hat{P}^0}{\partial t}(t, x(t)) + \frac{x(t)^2 \sigma^2(0)}{2} \frac{\partial \hat{P}^0}{\partial x^2}(t, x(t)) + r x(t) \frac{\partial \hat{P}^0}{\partial x}(t, x(t)) - r \hat{P}^0(t, x(t)) = 0,$$

where we have set $x(t) = xe^{R\lambda^*(T-t)}$. As $s \to \hat{P}^0(s, x)$ is non-increasing, $y \to \hat{P}^0(t, y)$ is non-increasing.
and convex, we have

\[
\frac{K^2 \sigma^2(0) e^{2R\lambda^*(T-t)}}{2} \frac{\partial^2 \check{P}^0}{\partial x^2}(t, x(t)) \geq \frac{x(t)^2 \sigma^2(0)}{2} \frac{\partial^2 \check{P}^0}{\partial x^2}(t, x(t)) \\
\geq -r x(t) \frac{\partial \check{P}^0}{\partial x}(t, x(t)) + r \check{P}^0(t, x(t)) \\
\geq -r x(t) \frac{\partial \check{P}^0}{\partial x}(t, x(t)) + r(K - x(t)) \\
\geq rK - r x(t) \left( \frac{\partial \check{P}^0}{\partial x}(t, x(t)) + 1 \right) \\
\geq rK - rK e^{R\lambda^*(T-t)} \left( \frac{\partial \check{P}^0}{\partial x}(t, x(t)) + 1 \right)
\]

Now, we integrate the last inequality between \( \hat{b}(t, 0) e^{-R\lambda^*(T-t)} \) and \( y \in (\hat{b}(t, 0) e^{-R\lambda^*(T-t)}, b(t, 0)) \) and use the smooth fit property to obtain

\[
\frac{K^2 \sigma^2(0) e^{R\lambda^*(T-t)}}{2} \left( \frac{\partial \check{P}^0}{\partial x}(t, y(t)) + 1 \right) \geq rK(y - \hat{b}(t, 0) e^{-R\lambda^*(T-t)}) - rK \left( \check{P}^0(t, y(t)) - f(y(t), 0) \right)
\]

where we have set \( y(t) = ye^{R\lambda^*(T-t)} \geq y \). It follows from the previous step that \( \check{P}^0(t, y(t)) - f(y, 0) = \check{P}^0(t, y(t)) - P(t, y, 0) \leq \lambda^*(c y + d) \sqrt{T - t} \); then we have

\[
\frac{K^2 \sigma^2(0) e^{R\lambda^*(T-t)}}{2} \left( \frac{\partial \check{P}^0}{\partial x}(t, y(t)) + 1 \right) \geq rK(y - \hat{b}(t, 0)) - rK \lambda^*(cK + d) \sqrt{T - t}.
\]

Once again, we integrate this inequality between \( \hat{b}(t, 0) e^{-R\lambda^*(T-t)} \) and \( b(t, 0) \) and get

\[
\frac{K^2 \sigma^2(0)}{2} \left( \check{P}^0(t, \hat{b}(t, 0) e^{R\lambda^*(T-t)}) - f(\hat{b}(t, 0) e^{R\lambda^*(T-t)}, 0) \right) \geq \frac{rK}{2} (\Delta b(t))^2 - rK \lambda^*(cK + d) \sqrt{T - t} \Delta b(t).
\]

where we have set \( \Delta b(t) = b(t, 0) - \hat{b}(t, 0) e^{-R\lambda^*(T-t)} \). From the previous step, we deduce that

\[
\frac{K^2 \sigma^2(0)}{2} \left( \lambda^*(cK + d) \sqrt{T - t} + K(e^{R\lambda^*(T-t)} - 1) \right) \geq \frac{rK}{2} (\Delta b(t))^2 - rK \lambda^*(cK + d) \sqrt{T - t} \Delta b(t).
\]

It implies that for \( \lambda^* \) going to 0, there exists \( C > 0 \) which does not depend on \( t \) such that

\[
0 \leq b(t, 0) - \hat{b}(t, 0) e^{-R\lambda^*(T-t)} \leq C \sqrt{\lambda^*} + o(\sqrt{\lambda^*}).
\]

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References


