Multipliers and Morrey spaces

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Abstract :

We study the pointwise multipliers from one Morrey space to another Morrey space. We give a necessary and sufficient condition to grant that the space of those multipliers is a Morrey space as well.

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Introduction.

This paper deals with multiplication by functions in Morrey spaces. For $1 < p < \infty$ and $0 \leq \lambda \leq d$, the Morrey space $\mathcal{L}_{p,\lambda}(\mathbb{R}^d)$ is defined as the space of locally integrable functions on $\mathbb{R}^d$ such that

$$\sup_{Q \in \mathcal{Q}} R_Q^{-\lambda} \int_Q |f(x)|^p \, dx < \infty \tag{1}$$

where $\mathcal{Q}$ is the collection of cubes Q and where R_Q is the size of $Q = x_Q + [0, R_Q]^d$.

The notation $\mathcal{L}_{p,\lambda}$ (used for instance by Peetre [8]) is not used by all authors. Vega uses $\mathcal{L}^{\alpha,p}$ with the condition that

$$\sup_{Q\in\mathcal{Q}} R_Q^\alpha\left(\frac{1}{|Q|}\int_Q |f(x)|^p\, dx\right)^{1/p} < \infty$$

where $|Q|$ is the Lebesgue measure of Q. We have $\mathcal{L}_{p,\lambda} = \mathcal{L}^{\alpha,p}$ with $\lambda = d - p\alpha$. We shall use another notation, the space $\dot{M}^{p,q}$ defined by

$$\sup_{Q\in\mathcal{Q}} R_Q^{d/q-d/p}\left(\int_Q |f(x)|^p\, dx\right)^{1/p} < \infty$$

We have $\dot{M}^{p,q} = \mathcal{L}^{\alpha,p}$ with $q = d/\alpha$.

The restrictions on λ, α or q are the following ones : $0 \le \lambda \le d$ (if f satisfies inequality (1) for $\lambda < 0$ or $\lambda > d$, then $f = 0$), and thus $0 \le \alpha \le d/p$ and $p \le q \le +\infty$. Moreover, we have :
– $\mathcal{L}_{p,0} = \mathcal{L}^{d/p,p} = \dot{M}^{p,p} = L^p$
– $\mathcal{L}_{p,d} = \mathcal{L}^{0,p} = \dot{M}^{p,\infty} = L^\infty$
– for $\alpha = d/q$, $L^q \subset \dot{M}^{p,q} = \mathcal{L}^{\alpha,p} \subset \dot{B}_\infty^{-\alpha,\infty}$.

We define $\|f\|_{\dot{M}^{p,q}} = \sup_{Q\in\mathcal{Q}} R_Q^{d/q-d/p}(\int_Q |f(x)|^p\, dx)^{1/p}$. We have the Hölder estimate : if $f \in \dot{M}^{p_0,q_0}$ and $g \in \dot{M}^{p_1,q_1}$ with $1/p = 1/p_0 + 1/p_1 < 1$ and $1/q = 1/q_0 + 1/q_1$, then $fg \in \dot{M}^{p,q}$ and $\|fg\|_{\dot{M}^{p,q}} \le \|f\|_{\dot{M}^{p_0,q_0}} \|g\|_{\dot{M}^{p_1,q_1}}$. The motivation of our paper is to study the reverse inequality : when do we have

$$\|g\|_{\dot{M}^{p_1,q_1}} \le C \sup_{\|f\|_{M^{p_0,q_0}}\le 1} \|fg\|_{\dot{M}^{p,q}} \quad ?$$

As we shall see, a necessary and sufficient condition to get this reverse inequality is that $q_1/p_1 \ge q_0/p_0$ (or, equivalently, if $\dot{M}^{p_i,q_i} = \mathcal{L}_{p,\lambda_i}$, that $\lambda_1 \ge \lambda_0$). In the case $\lambda_1 < \lambda_0$, we construct a counter-example based on a fractal set K^β with Hausdorff dimension $\beta = d - d\frac{p_1}{q_1}$ (or $\beta = d - p_1\alpha_1 = \lambda_1$). This fractal set will allow us to recover simple counterexamples for trace inequalities or interpolation of operators.

1 Statement of the results.

We first consider the problem of pointwise multipliers between Morrey spaces. Our result is the following one :

Theorem 1:

Let $1 < p \leq q$ and $1 < p_0 \leq q_0 < \infty$. Let $\mathcal{M}(\dot{M}^{p_0,q_0} \to \dot{M}^{p,q})$ be the set of pointwise multipliers from $\dot{M}^{p_0,q_0}$ to $\dot{M}^{p,q}$, with norm

$$\|f\|_{\mathcal{M}(\dot{M}^{p_0,q_0} \to \dot{M}^{p,q})} = \sup_{\|g\|_{\dot{M}^{p_0,q_0}} \leq 1} \|fg\|_{\dot{M}^{p,q}}$$

Then :
i) $\mathcal{M}(\dot{M}^{p_0,q_0} \to \dot{M}^{p,q}) \neq \{0\}$ *if and only if* $p \leq p_0$ *and* $q \leq q_0$.
ii) If $p \leq p_0$ $(1/p = 1/p_0 + 1/p_1)$ *and* $q \leq q_0$ $(1/q = 1/q_0 + 1/q_1)$*, then we have the embeddings* $\dot{M}^{p_1,q_1} \subset \mathcal{M}(\dot{M}^{p_0,q_0} \to \dot{M}^{p,q}) \subset \dot{M}^{p,q_1}$
iii) $\dot{M}^{p_1,q_1} = \mathcal{M}(\dot{M}^{p_0,q_0} \to \dot{M}^{p,q})$ *if and only if* $q_1/p_1 \geq q_0/p_0$*. In this case, we have equality of norms.*

We shall next consider the problem of trace inequalities. We will show that the limit case is not fulfilled for the Fefferman–Phong inequality [3] :

Theorem 2

Let $1 < p < \infty$ and $0 < r < d/p$. Let $\mathcal{M}(\dot{W}^{r,p} \to \dot{L}^p)$ be the set of pointwise multipliers from $\dot{W}^{r,p}$ to L^p, with norm

$$\|f\|_{\mathcal{M}(\dot{W}^{r,p} \to L^p)} = \sup_{\|g\|_p \leq 1} \|fI^r g\|_p$$

Then :
i) If $p < p_1$*, then we have the embeddings* $\dot{M}^{p_1,d/r} \subset \mathcal{M}(\dot{W}^{r,p} \to \dot{L}^p) \subset \dot{M}^{p,d/r}$
ii) $\mathcal{M}(\dot{W}^{r,p} \to \dot{L}^p) \neq \dot{M}^{p,d/r}$

We shall end with Ruiz and Vega's counterexample for interpolation [11] and give a counterexample for every case when interpolation fails :

Theorem 3

Let $1 < p_0 \leq q_0 < \infty$ and $1 < p_1 \leq q_1 < \infty$. Let $0 < \theta < 1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$. Then :
i) We have $[\dot{M}^{p_0,q_0}, \dot{M}^{p_1,q_1}]_\theta \subset \dot{M}^{p,q}$ *(complex interpolation)*
ii) $[\dot{M}^{p_0,q_0}, \dot{M}^{p_1,q_1}]_\theta = \dot{M}^{p,q}$ *(with equivalence of norms) if and only if* $p_0/q_0 = p_1/q_1$.
iii) We have $[\dot{M}^{p_0,q_0}, \dot{M}^{p_1,q_1}]_{\theta,p} \subset \dot{M}^{p,q}$ *(real interpolation)*

iv) $\dot{M}^{p,q} \subset [\dot{M}^{p_0,q_0}, \dot{M}^{p_1,q_1}]_{\theta,\infty}$ *(continuous embedding) if and only if* $p_0/q_0 = p_1/q_1$.
v) $[\dot{M}^{p_0,q_0}, \dot{M}^{p_1,q_1}]_{\theta,\infty} \subset \dot{M}^{p,q}$ *if and only if* $p_0 = p_1$

2 The fractal set K^β

In this section we construct a fractal subset K^β of $\mathbb{R}^d$ (with Hausdorff dimension $\beta \in [0, d)$). We shall use the dyadic cubes $Q_{j,k} = \prod_{i=1}^d [k_i 2^{-j}, (k_i+1)2^{-j}]$:

$$\mathbf{1}_{Q_{j,k}}(x) = \mathbf{1}_{[0,1]^d}(2^j x - k).$$

We inductively define $K_j^\beta = \cup_{k \in K_j} Q_{j,k}$ in the following way :
i) $K_0^\beta = Q_{0,0} = [0,1]^d$.
ii) Let $K_j^\beta = \cup_{k \in K_j} Q_{j,k}$. If $\#(K_j) \leq 2^{(j+1)\beta - d}$, then we keep in each $Q_{j,k}$, $k \in K_j$, the 2^d dyadic cubes of size $2^{-(j+1)}$ contained in $Q_{j,k}$, so that $K_{j+1}^\beta = K_j^\beta$ and $\#(K_{j+1}) = 2^d\ \#(K_j)$.
iii) Let $K_j^\beta = \cup_{k \in K_j} Q_{j,k}$. If $\#(K_j) > 2^{(j+1)\beta - d}$, then we keep in each $Q_{j,k}$, $k \in K_j$, only one dyadic cube $Q_{j+1,2k}$ of size $2^{-(j+1)}$ contained in $Q_{j,k}$, so that $K_{j+1}^\beta \subset K_j^\beta$ and $\#(K_{j+1}) = \#(K_j)$.

By induction, we see that $2^{j\beta - d} < \#(K_j) \leq 2^{j\beta}$:
– $\#(K_0) = 1 = 2^{0\beta}$;
– in case ii) we have $2^{(j+1)\beta - d} = 2^{j\beta - d} 2^\beta < 2^d\ \#(K_j) = \#(K_{j+1}) \leq 2^d 2^{(j+1)\beta - d} = 2^{(j+1)\beta}$
– in case iii) we have $2^{(j+1)\beta - d} < \#(K_j) = \#(K_{j+1}) \leq 2^{j\beta} \leq 2^{(j+1)\beta}$

If $Q = Q_{j,k}$ is a dyadic cube contained in $[0,1]^d$ such that $|Q \cap K_n^\beta| > 0$, then we have $|Q \cap K_n^\beta| = |Q|$ if $j \geq n$ and $|Q \cap K_n^\beta| = \frac{\#(K_n)}{\#(K_j)} 2^{-nd}$ if $j \leq n$ (with $2^{-d} 2^{(n-j)\beta} \leq \frac{\#(K_n)}{\#(K_j)} \leq 2^d 2^{(n-j)\beta}$).

The next step is to introduce the measures $\mu_n^\beta = \frac{1}{|K_n^\beta|} \mathbf{1}_{K_n^\beta}\, dx$. This is a sequence of probability measures and we may find a subsequence $(\mu_{n_k}^\beta)_{k \in \mathbb{N}}$ which converges vaguely to a probability measure μ^β. This measure μ^β is supported by the compact set $K^\beta = \cap_{n \in \mathbb{N}} K_n^\beta$.

If $Q = Q_{j,k}$ is a dyadic cube contained in $[0,1]^d$, such that $\mu^\beta(Q) > 0$, then we have $\mu^\beta(Q) = \frac{1}{\#(K_j)}$ and thus $2^{-j\beta} \leq \mu^\beta(Q) \leq 2^d 2^{-j\beta}$. Thus, we find that K^β has Hausdorff dimension β and that $0 < \mathcal{H}^\beta(K^\beta) < +\infty$.

A classical result of potential theory [7] [1] then states that the Riesz potential $I^\alpha \mu^\beta$ (convolution of μ^β with a kernel $k^\alpha(x) = c_{\alpha,d}\frac{1}{|x|^{d-\alpha}}$) satisfies the following equality :

Lemma 1 :
For $1 < p < \infty$, $0 < \alpha \leq d/p$ and $\beta = d - p\alpha$,

$$\int (I_\alpha \mu^\beta)^{\frac{p}{p-1}} \, dx = +\infty \tag{2}$$

Proof : We have $\mathcal{F}(I^\alpha \mu) = |\xi|^{-\alpha}\hat{\mu}$. Thus, if Λ^α is the operator defined by $\mathcal{F}(\Lambda^\alpha \varphi) = |\xi|^\alpha \hat{\varphi}$, we have $\int \varphi d\mu^\beta = \int I^\alpha \mu^\beta \Lambda^\alpha \varphi \, dx$. We choose $\omega \in \mathcal{D}$ such that $\mathbf{1}_{[0,1]^d} \leq \omega$. We then define

$$\omega_j(x) = \sum_{k \in K_j} \omega(2^j x - k)$$

Since $\omega_j \geq \mathbf{1}_{K_j^\beta} \geq \mathbf{1}_{K^\beta}$, we find that

$$1 = \mu^\beta(K^\beta) \leq \int I^\alpha \mu^\beta \Lambda^\alpha \omega_j \, dx \tag{3}$$

We now estimate the size and decay of

$$\Lambda^\alpha(\sum_{k \in \mathbb{Z}^d} \lambda_k \omega(x-k)) :$$

i) for $\gamma \in \mathbb{N}^d$, we have obviously

$$\|\sum_{k \in \mathbb{Z}^d} \lambda_k \partial^\gamma \omega(x-k))\|_p \leq C_{\gamma,\omega}(\sum_{k \in \mathbb{Z}^d} |\lambda_k|^p)^{1/p}$$

ii) by interpolation, we get

$$\|\Lambda^\alpha(\sum_{k \in \mathbb{Z}^d} \lambda_k \omega(x-k)) \leq C_{\alpha,\omega}(\sum_{k \in \mathbb{Z}^d} |\lambda_k|^p)^{1/p} \tag{4}$$

iii) Since Λ^α is a convolution with a distribution which is equal to $C_{\alpha,d}\frac{1}{|x|^{d+\alpha}}$, we find that, for R larger than 2δ where δ is the diameter of the support of ω,

$$\sum_{k\in\mathbb{Z}^d} \mathbf{1}_{|x-k|>R}|\lambda_k\Lambda^\alpha\omega(x-k)| \le C_{\alpha,d}\int_{|x-y|>R/2}\frac{1}{|x-y|^{d+\alpha}}\sum_{k\in\mathbb{Z}^d}|\lambda_k||\omega(y)|\,dy$$

so that

$$\|\sum_{k\in\mathbb{Z}^d}\mathbf{1}_{|x-k|>R}|\lambda_k\Lambda^\alpha\omega(x-k)|\|_p \le C_{\alpha,\omega}R^{-\alpha}(\sum_{k\in\mathbb{Z}^d}|\lambda_k|^p)^{1/p} \tag{5}$$

Now, assume that $I^\alpha\mu^\beta\in L^{\frac{p}{p-1}}$. From (3), we would get

$$1\le \|I^\alpha\mu^\beta\|_{\frac{p}{p-1}}\|\mathbf{1}_{d(x,K_j^\beta)>R2^{-j}}\Lambda^\alpha\omega_j\|_p+\|\mathbf{1}_{d(x,K_j^\beta)\le R2^{-j}}I^\alpha\mu^\beta\|_{\frac{p}{p-1}}\|\Lambda^\alpha\omega_j\|_p = A_{j,R}+B_{j,R}.$$

The control of $A_{j,R}$ is given by (5) :

$$A_{j,R}\le CR^{-\alpha}\|I^\alpha\mu^\beta\|_{\frac{p}{p-1}}2^{j\alpha-j\frac{d}{p}}(\#(K_j))^{1/p}\le CR^{-\alpha}\|I^\alpha\mu^\beta\|_{\frac{p}{p-1}}$$

Thus, we could choose $R>0$ such that : $\sup_{j\in\mathbb{N}}A_{j,R}<1/2$. Now, we control $B_{j,R}$ through (4) :

$$B_{j,R}\le C\|\mathbf{1}_{d(x,K_j^\beta)\le R2^{-j}}I^\alpha\mu^\beta\|_{\frac{p}{p-1}}2^{j\alpha-j\frac{d}{p}}(\#(K_j))^{1/p}\le C\|\mathbf{1}_{d(x,K_j^\beta)\le R2^{-j}}I^\alpha\mu^\beta\|_{\frac{p}{p-1}}$$

Since

$$|\{x\in\mathbb{R}^d\ /d(x,K_j^\beta)\le R2^{-j}\}|\le C\#(K_j)(1+R)^d2^{-jd}\le C(1+R)^d2^{j(\beta-d)},$$

we find that $\lim_{j\to+\infty}B_{j,R}=0$. This would give $1<1/2\ldots$ ◇

Lemma 1 has the following corollary :

Corollary 1 :
For $1<p<\infty$, $0<\alpha\le d/p$ and $\beta=d-p\alpha$,

$$\sup_{n\in\mathbb{N}}\int(I_\alpha\mu_n^\beta)^{\frac{p}{p-1}}\,dx=+\infty \tag{6}$$

The sets are K_n^β are very interesting for generating examples and counterexamples in Morrey spaces. Indeed, we have the following result :

Lemma 2 :
Let (α_n) *be a sequence of non-negative numbers. Let* $f = \sum_{n\in\mathbb{N}} \alpha_n \mathbf{1}_{K_n^\beta}$. *Then the following statements are equivalent :*
i) $f \in L^p$
ii) $\sum_{n\in\mathbb{N}} \alpha_n^p 2^{n(\beta-d)} < \infty$
If moreover $p \le q \le \frac{pd}{d-\beta}$*, then i) and ii) are equivalent to*
iii) $f \in \dot{M}^{p,q}$

Proof : *i)* $\Rightarrow$ *ii)* is obvious, since $\sum_{n\in\mathbb{N}} \alpha_n^p \mathbf{1}_{K_n^\beta} = \sum_{n\in\mathbb{N}} \alpha_n^p \mathbf{1}_{K_n^\beta}^p \le (\sum_{n\in\mathbb{N}} \alpha_n \mathbf{1}_{K_n^\beta})^p$.

Conversely, we have

$$\int |\sum_{n\in\mathbb{N}} \alpha_n \mathbf{1}_{K_n^\beta}|^p \, dx \le \int \sum_{n\in\mathbb{N}} (\sum_{k=0}^n \alpha_k)^p \mathbf{1}_{K_n^\beta} \, dx \le \sum_{n\in\mathbb{N}} (\sum_{k=0}^n \alpha_k)^p 2^{n(\beta-d)}$$

hence

$$\int |\sum_{n\in\mathbb{N}} \alpha_n \mathbf{1}_{K_n^\beta}|^p \, dx \le \sum_{n\in\mathbb{N}} (\sum_{k=0}^n \alpha_k 2^{k(\beta-d)/p} 2^{(n-k)(\beta-d)/p})^p$$

and finally

$$\int |\sum_{n\in\mathbb{N}} \alpha_n \mathbf{1}_{K_n^\beta}|^p \, dx \le (\sum_{k=0}^\infty \alpha_k^p 2^{k(\beta-d)}) \, (\sum_{k=0}^\infty 2^{-(\beta-d)/p})^p.$$

So that *ii)* $\Rightarrow$ *i)*.

Since f is supported in $[0,1]^d$, we have, for $p \le q \le \frac{dp}{d-\beta}$, $f \in \dot{M}^{p,\frac{pd}{d-\beta}} \Rightarrow f \in \dot{M}^{p,q} \Rightarrow f \in L^p$.

We now prove $f \in L^p \Rightarrow f \in \dot{M}^{p,\frac{pd}{d-\beta}}$. It is enough to estimate the norm $\|f\mathbf{1}_{Q_{j,k}}\|_p$ for a dyadic cube (with $j \ge 0$). We write $f_j = \sum_{n<j} \alpha_n \mathbf{1}_{K_n^\beta}$ and $g_j = f - f_j$. We have

$$\int_{Q_{j,k}} |g_j|^p \, dx \le \int_{Q_{j,k}} \sum_{n\ge j} (\sum_{k=j}^n \alpha_k)^p \mathbf{1}_{K_n^\beta} \, dx \le 2^d \sum_{n\in\mathbb{N}} (\sum_{k=j}^n \alpha_k)^p 2^{(n-j)\beta} 2^{-nd}$$

hence

$$\int_{Q_{j,k}} |g_j|^p \, dx \le C\|f\|_p^p 2^{-j\beta}.$$

On the other hand, we have

$$\int_{Q_{j,k}} |f_j|^p \, dx \le 2^{-jd}(\sum_{n<j} \alpha_n)^p \le C\|f\|_p^p 2^{-jd}(\sum_{n<j} 2^{\frac{n(d-\beta)}{p-1}})^{p-1} \le C'\|f\|_p^p 2^{-j\beta}.$$

Thus, we find that $f \in L^p \Rightarrow f \in \mathcal{L}^{p,\beta} = \dot{M}^{p,\frac{dp}{d-\beta}}$. ◇

3 Pointwise products.

In this section, we prove theorem 1. Let $1 < p \le q$ and $1 < p_0 \le q_0 < \infty$. We study the space $X = \mathcal{M}(\dot{M}^{p_0,q_0} \to \dot{M}^{p,q})$.

a) Case $q > q_0$ **:** Let Q be a cube. Then we have, for $f \in X$,

$$\int_Q |f|^p \, dx \le \|f\mathbf{1}_Q\|^p_{\dot{M}^{p,q}} |Q|^{1-\frac{p}{q}} \le \|f\|_X^p \|\mathbf{1}_Q\|^p_{\dot{M}^{p_0,q_0}} |Q|^{1-\frac{p}{q}} \le \|f\|_X^p |Q|^{1-p/q+p/q_0}$$

Hence, $f \in \mathcal{L}^{p,d(1-p/q+p/q_0)}$. If $q > q_0$, $d(1-p/q+p/q_0) > d$, hence $\mathcal{L}^{p,d(1-p/q+p/q_0)} = \{0\}$.

b) Case $p > p_0$ **:** If $f \in X$ and $\varphi \in \mathcal{D}$, then $f * \varphi \in X$:

$$\|(f*\varphi)g\|_{\dot{M}^{p,q}} \le \int |\varphi(y)| \|f(x-y)g(x)\|_{\dot{M}^{p,q}} \, dy \le \|\varphi\|_1 \|f\|_X \|g\|_{\dot{M}^{p_0,q_0}}$$

If $f * \varphi \neq 0$, we may find $\gamma > 0$ and a cube Q such that $\gamma|f*\varphi| \ge \mathbf{1}_Q$. This proves that $\mathbf{1}_Q \in X$. In that case, we would have $\dot{M}^{p_0,q_0} \subset L^p_{loc}$. But this is false if $p > p_0$: take for instance $f = \sum_{n\ge1} \frac{1}{n} 2^{\frac{n(d-\beta)}{p_0}} \mathbf{1}_{K_n^\beta}$, with $\beta = d - \frac{dp_0}{q_0}$. Using lemma 2, we find that $f \in M^{p_0,q_0}$ but $f \notin L^p$.

c) Case $p \le p_0$ **and** $q \le q_0$ **:** In that case, we have $X \neq \{0\}$. For instance, $\mathbf{1}_{[0,1]^d} \in X$. If $g \in \dot{M}^{p_0,q_0}$, then $g \in \dot{M}^{p,q_0}$, thus $\mathbf{1}_{[0,1]^d} g \in \dot{M}^{p,q_0} \cap L^p = \dot{M}^{p,q_0} \cap \dot{M}^{p,p} \subset \dot{M}^{p,q}$.

d) Easy embeddings : Let $p \le p_0$ and $q \le q_0$. We write $1/p = 1/p_0 + 1/p_1$ and $1/q = 1/q_0 + 1/q_1$. We have seen (when discussing the case $q > q_0$) that $X \subset \mathcal{L}^{p,d(1-p/q+p/q_0)} = \dot{M}^{p,q_1}$. Moreover, Hölder inequality gives us easily that $\dot{M}^{p_1,q_1} \subset X$.

e) Case $q_1/p_1 \geq q_0/p_0$ **:** If $q_0 = q$, hence $q_1 = +\infty$, we have $L^\infty = \dot{M}^{p_1,q_1} \subset X \subset \dot{M}^{p,q_1} = L^\infty$. Thus, we consider only the case $q < q_0$. (Thus, $q_1 < +\infty$, hence $p_1 < +\infty$). If $f \in X$, then $f_R = f\mathbf{1}_{B(0,R)}\mathbf{1}_{\{x\ /\ |f(x)|<R\}} \in X$ and $\|f_R\|_X \leq \|f\|_X$. Moreover, $\|f\|_{\dot{M}^{p_1,q_1}} = \sup_{R>0} \|f_R\|_{\dot{M}^{p_1,q_1}}$. Thus, it is enough to prove that $\|f\|_{\dot{M}^{p_1,q_1}} \leq C\|f\|_X$ for $f \in L^1 \cap L^\infty \subset \dot{M}^{p_1,q_1}$. For a cube Q, we write :

$$\int_Q |f|^{p_1}\, dx = \int_Q |f|f|^{\frac{p_1-p}{p}}\mathbf{1}_Q|^p\, dx \leq \|f\|_X^p \|\mathbf{1}_Q|f|^{\frac{p_1-p}{p}}\|^p_{M^{p_0,q_0}}|Q|^{1-\frac{p}{q}}$$

and we thus have

$$\int_Q |f|^{p_1}\, dx \leq \|f\|_X^p \|\mathbf{1}_Q f\|^{p_1-p}_{M^{p_0\frac{p_1-p}{p},q_0\frac{p_1-p}{p}}}|Q|^{1-\frac{p}{q}} = \|f\|_X^p \|\mathbf{1}_Q f\|^{p_1-p}_{M^{p_1,q_0\frac{p_1}{p_0}}}|Q|^{1-\frac{p}{q}}$$

Since $q_0\frac{p_1}{p_0} \leq q_1$, we have $\|\mathbf{1}_Q f\|_{M^{p_1,q_0\frac{p_1}{p_0}}} \leq |Q|^{-\frac{1}{q_1}+\frac{p_0}{p_1 q_0}}\|\mathbf{1}_Q f\|_{M^{p_1,q_1}}$ and thus

$$\int_Q |f|^{p_1}\, dx \leq \|f\|_X^p \|f\|^{p_1-p}_{M^{p_1,q_1}}|Q|^{1-\frac{p}{q}+(p_1-p)(-\frac{1}{q_1}+\frac{p_0}{p_1 q_0})} = \|f\|_X^p \|f\|^{p_1-p}_{M^{p_1,q_1}}|Q|^{1-\frac{p_1}{q_1}}$$

Thus, $\|f\|^{p_1}_{\dot{M}^{p_1,q_1}} \leq \|f\|_X^p \|f\|^{p_1-p}_{M^{p_1,q_1}}$, and we may conclude that $\|f\|_{\dot{M}^{p_1,q_1}} \leq \|f\|_X$. Since the reverse inequality is obvious, we find that $\|f\|_{\dot{M}^{p_1,q_1}} = \|f\|_X$.

f) Case $q_1/p_1 < q_0/p_0$ **:** If $q_1 < p_1$, then $\dot{M}^{p_1,q_1} = \{0\} \neq X$. Thus, we now consider the case $p_1 \leq q_1 < p_1\frac{q_0}{p_0}$. Let $f = \sum_{k\in\mathbb{N}} 2^{\frac{n(d-\beta)}{p_1}}\mathbf{1}_{K_n^\beta}$ with $\beta = d - \frac{dp_1}{q_1}$. From Lemma 2, we know that $f \notin \dot{M}^{p_1,q_1}$. We shall prove that $f \in X$. Let $g \in \dot{M}^{p_0,q_0}$. We estimate $\|fg\mathbf{1}_Q\|_p$ for a cube Q. Since f is supported in $[0,1]^d$, it is enough to take dyadic cubes $Q_{j,k}$ with $j \geq 0$. On $K_n^\beta - K_{n+1}^\beta$ we have $2^{\frac{n(d-\beta)}{p_1}} \leq f \leq C2^{\frac{n(d-\beta)}{p_1}}$, so that

$$\int_{Q_{j,k}} |fg|^p\, dx \leq C\sum_{n\in\mathbb{N}} 2^{\frac{n(d-\beta)p}{p_1}} \int_{Q_{j,k}\cap K_n^\beta} |g|^p\, dx$$

When $n < j$, we write

$$\int_{Q_{j,k}\cap K_n^\beta} |g|^p\, dx \leq \int_{Q_{j,k}} |g|^p\, dx \leq \|g\|^p_{\dot{M}^{p_0,q_0}} 2^{-jd(1-\frac{p}{q_0})}$$

When $n \geq j$, we write

$$\int_{Q_{j,k}\cap K_n^\beta} |g|^p\, dx = \sum_{l\in K_n} \int_{Q_{j,k}\cap Q_{n,l}} |g|^p\, dx \leq \frac{\#(K_n)}{\#(K_j)}\|g\|^p_{\dot{M}^{p_0,q_0}} 2^{-nd(1-\frac{p}{q_0})}$$

Thus, we get

$$\int_{Q_{j,k}} |fg|^p \, dx \le C\|g\|^p_{\dot M^{p_0,q_0}} \Big(\sum_{n<j} 2^{-jd(1-\frac{p}{q_0})} 2^{nd\frac{p}{q_1}} + \sum_{n\ge j} 2^{(n-j)d(1-\frac{p_1}{q_1})} 2^{-nd(1-\frac{p}{q_0})} 2^{nd\frac{p}{q_1}} \Big)$$

or, equivalently,

$$\int_{Q_{j,k}} |fg|^p \, dx \le C\|g\|^p_{\dot M^{p_0,q_0}} 2^{-jd(1-\frac{p}{q})} \Big(\sum_{n<j} 2^{(n-j)d\frac{p}{q_1}} + \sum_{n\ge j} 2^{(n-j)d(\frac{p}{q}-\frac{p_1}{q_1})} \Big)$$

We have

$$\frac{1}{p} = \frac{1}{p_0} + \frac{1}{p_1} > \frac{q_1}{p_1 q_0} + \frac{1}{p_1} = \frac{q_1}{p_1}\frac{1}{q}$$

so that $p/q - p_1/q_1 < 0$. This gives

$$\int_{Q_{j,k}} |fg|^p \, dx \le C\|g\|^p_{\dot M^{p_0,q_0}} 2^{-jd(1-\frac{p}{q})}$$

and thus $f \in X$.

4 Trace inequalities.

In [3], Fefferman states a theorem of Fefferman and Phong : for $s < d/2$, $f \in \dot H^s$ and $g \in \dot M^{p,d/s}$ with $2 < p \le d/s$, we have :

$$\|fg\|_2 \le C\|g\|_{\dot M^{p,d/s}} \|f\|_{\dot H^s} \tag{7}$$

Thus, belonging to the Morrey space $\dot M^{p,d/s}$ with $p > 2$ is a sufficient condition for belonging to the space of multipliers from $\dot H^s$ to L^2. The space of such multipliers has been studied by several authors, including Maz'ya [5] [6].

More generally, trace inequalities deals with nonnegative measures μ such that

$$\int |I^\alpha f|^p \, d\mu \le C \int |f|^p \, dx \tag{8}$$

A necessary condition on μ has been given by Kermann and Sawyer in [4]. It is now well known that, for $d\mu = |g|^p \, dx$, a sufficient condition on g for (8) to hold (with $1 < p < d/\alpha$) is that $g \in \dot M^{p_1,d/\alpha}$ for some $p < p_1 \le d/\alpha$ (see [9] for instance for further references). Thus, if $1 < p < \infty$ and $0 < r < d/p$, if moreover $p < p_1 \le d/r$, then, for $f \in L^p$ and $g \in \dot M^{p_1,d/r}$ we have

$$\|gI_r f\|_r \le C\|f\|_{\dot M^{p,q}} \|g\|_r \tag{9}$$

In this section, we shall check that (9) is no longer valid when $p_1 = d/r$. This has been known for long when $pr = k \in \mathbb{N}$ (by considering functions $\phi(x_1, \ldots, x_d) = \phi(x_1, \ldots, x_k)$ with $\phi \in L^p(\mathbb{R}^k)$). Counterexamples have been recently given by Qixiang Yang [12] for any $r \in (0, d/p)$.

Our counterexample is slightly different from Yang's example, and is based on our sets K_n^β, with $\beta = d - pr$. If $g \in \mathcal{M}(\dot{W}^{r,p} \to L^p) = \dot{X}^{r,p}$, we have by duality that, for all $f \in L^{\frac{p}{p-1}}$

$$(10) \qquad \|I_r(fg)\|_{\frac{p}{p-1}} \leq C\|g\|_{\dot{X}^{r,p}} \|f\|_{\frac{p}{p-1}}$$

We are going to exhibit $f \in L^{\frac{p}{p-1}}$ and $g \in \dot{M}^{p,d/r}$ such that $\|I_r(fg)\|_{\frac{p}{p-1}} = +\infty$. Using corollary 1, we take $\beta = d - rp$ fix an increasing sequence $(n_k)_{k\in\mathbb{N}^*}$ of integers such that $\|I_r \mu_{n_k}^\beta\|_{\frac{p}{p-1}} \geq k^3$. We define $f = \sum_{k\in\mathbb{N}^*} \frac{1}{k} 2^{n_k(d-\beta)\frac{p-1}{p}} \mathbf{1}_{K_{n_k}^\beta}$ and $g = \sum_{k\in\mathbb{N}^*} \frac{1}{k} 2^{n_k(d-\beta)\frac{1}{p}} \mathbf{1}_{K_{n_k}^\beta}$. From Lemma 2, we get that $f \in L^{\frac{p}{p-1}}$ and $g \in \dot{M}^{p,d/r}$. Moreover $fg \geq \frac{1}{k^2} 2^{n_k(d-\beta)} \mathbf{1}_{K_{n_k}^\beta} \geq 2^{-d} \frac{1}{k^2} \mu_{n_k}^\beta$. Hence, $\|I_r(fg)\|_{\frac{p}{p-1}} \geq k$ for every k , and thus $\|I_r(fg)\|_{\frac{p}{p-1}} = +\infty$. ◇

5 Non-interpolation results.

Let $1 < p_0 \leq q_0 < \infty$ and $1 < p_1 \leq q_1 < \infty$. Let $0 < \theta < 1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$. If F is an interpolation functor such that, for any Banach pair (A_0, A_1) and any bounded operator T from A_0 to L^{p_0} (with operator norm M_0) and A_1 to L^{p_1} (with operator norm M_1) , then T is bounded from $F(A_0, A_1)$ to L^p with operator norm $M \leq C M_0^{1-\theta} M_1^\theta$ (where the constant C does not depend on T), then it is easy to see that $F(\dot{M}^{p_0,q_0}, \dot{M}^{p_1,q_1}) \subset \dot{M}^{p,q}$ [10] : it is enough to interpolate the operator norms of $T_Q : f \mapsto \mathbf{1}_Q f$.

Thus, it is obvious that $[\dot{M}^{p_0,q_0}, \dot{M}^{p_1,q_1}]_\theta \subset \dot{M}^{p,q}$ (complex interpolation functor), $[\dot{M}^{p_0,q_0}, \dot{M}^{p_1,q_1}]_{\theta,p} \subset \dot{M}^{p,q}$ (real interpolation functor). Similarly, when $p_0 = p_1 = p$, we have $[\dot{M}^{p,q_0}, \dot{M}^{p,q_1}]_{\theta,\infty} \subset \dot{M}^{p,q}$ (real interpolation functor) .

Conversely, when $p_0/q_0 = p_1/q_1 = p/q$ we may define for $f \in \dot{M}^{p,q}$ the function $F(z) = \frac{f}{|f|} |f|^{(1-z)\frac{p}{p_0} + z\frac{p}{p_1}}$. This is a bounded continuous function of $z = x + iy$ (for $0 \leq x \leq 1$) with values in $\dot{M}^{p_0,q_0} + \dot{M}^{p_1,q_1}$, holomorphic on the strip $0 < x < 1$, with $\sup_{\in\mathbb{R}} \|F(iy)\|_{\dot{M}^{p_0,q_0}} < +\infty$, $\sup_{\in\mathbb{R}} \|F(1+iy)\|_{\dot{M}^{p_1,q_1}} <$

$+\infty$, and $F(\theta) = f$. This proves that $\dot{M}^{p,q} \subset [\dot{M}^{p_0,q_0}, \dot{M}^{p_1,q_1}]_\theta$ (complex interpolation functor).

We shall now give our counterexamples :

a) Non-inclusion of $\dot{M}^{p,q}$ into $[\dot{M}^{p_0,q_0}, \dot{M}^{p_1,q_1}]_{\theta,\infty}$ when $p_0/q_0 \neq p_1/q_1$:

By a duality argument, it is easy to see that a Banach B is continuously embedded into $[\dot{M}^{p_0,q_0}, \dot{M}^{p_1,q_1}]_{\theta,\infty}$ if and only the followingg assertion is true : for every linear form T which is bounded from $\dot{M}^{p_0,q_0}$ to $\mathbb{R}$ (with operator norm M_0) and bounded from $\dot{M}^{p_1,q_1}$ to $\mathbb{R}$ (with operator norm M_1), T is bounded from B to $\mathbb{R}$ and its operator norm M is bounded by $C_0 M_0^{1-\theta} M_1^\theta$ where the constant C_0 does not depend on T.

Thus, we shall follow the strategy of [11] and exhibit a sequence of linear forms T_n such that $\sup_{n\in\mathbb{N}} \frac{\|T_n\|_{\mathcal{L}(\dot{M}^{p,q}\to\mathbb{R})}}{\|T_n\|^{1-\theta}_{\mathcal{L}(\dot{M}^{p_0,q_0}\to\mathbb{R})}\|T_n\|^{\theta}_{\mathcal{L}(\dot{M}^{p_1,q_1}\to\mathbb{R})}} = +\infty$.

Our example is very simple : we just take $T_n(f) = \int_{K_n^\beta} f\ dx$ with $\beta = d(1-p/q)$. Our task is to estimate $\|T_n\|_{\mathcal{L}(\dot{M}^{r,s}\to\mathbb{R})}$ for $(r,s) = (p_0,q_0)$, (p,q) and (p_1,q_1).

We have

$$|K_n^\beta| = \#(K_n)2^{-nd} = T_n(\mathbf{1}_{K_n^\beta}) \leq \|T_n\|_{\mathcal{L}(\dot{M}^{p,q}\to\mathbb{R})}\|\mathbf{1}_{K_n^\beta}\|_{\dot{M}^{p,q}}$$

From Lemma 2, we see that

$$\|\mathbf{1}_{K_n^\beta}\|_{\dot{M}^{p,q}} \leq C2^{-d/q} = C2^{-n(d-\beta)/p},$$

hence

$$\|T_n\|_{\mathcal{L}(\dot{M}^{p,q}\to\mathbb{R})} \geq C2^{-n(d-\beta)(1-1/p)}$$

On the other hand, we have

$$|T_n(f)| \leq |K_n^\beta|^{1-1/p_i}(\int_{[0,1]^d} |f|^{p_i}\ dx)^{1/p_i} \leq C2^{-n(d-\beta)(1-1/p_i)}\|f\|_{\dot{M}^{p_i,q_i}}$$

and

$$|T_n(f)| \leq \sum_{k\in K_n}\int_{Q_{n,k}} |f|\ dx \leq C2^{n\beta}\|f\|_{\dot{M}^{p_i,q_i}}2^{-nd(1-1/q_i)}.$$

Thus, we have :

$$\|T_n\|_{\mathcal{L}(\dot{M}^{p_i,q_i}\to\mathbb{R})} \leq C2^{-n(d-\beta)(1-1/p_i)}\min(1, 2^{n(\beta/p_i+d/q_i-d/p_i)})$$

If $p_0/q_0 < p_1/q_1$ (hence $p_0/q_0 < p/q < p_1/q_1$), we write

$$d/q_i - (d-\beta)/p_i = d/q_i - dp/qp_i = d(p_i/q_i - p/q)/p_i$$

and find that

$$\|T_n\|^{1-\theta}_{\mathcal{L}(\dot M^{p_0,q_0}\to\mathbb{R})}\|T_n\|^{\theta}_{\mathcal{L}(\dot M^{p_1,q_1}\to\mathbb{R})} \le C2^{-n(d-\beta)(1-1/p)}2^{n(1-\theta)d(p_0/q_0-p/q)/p_0}.$$

Since $(1-\theta)d(p_0/q_0-p/q)/p_0 < 0$, we get that $\sup_{n\in\mathbb{N}} \frac{\|T_n\|_{\mathcal{L}(\dot M^{p,q}\to\mathbb{R})}}{\|T_n\|^{1-\theta}_{\mathcal{L}(\dot M^{p_0,q_0}\to\mathbb{R})}\|T_n\|^{\theta}_{\mathcal{L}(\dot M^{p_1,q_1}\to\mathbb{R})}} = +\infty$.

b) Non-inclusion of $[\dot M^{p_0,q_0},\dot M^{p_1,q_1}]_{\theta,\infty}$ into $\dot M^{p,q}$ when $p_0\neq p_1$:

We may assume $p_0<p_1$. Let $f=\sum_{n\in\mathbb{N}} 2^{\frac{n(d-\beta)}{p}}\mathbf{1}_{K_n^\beta}$ where

$$d>\beta\ge d\max(1-\frac{p_0}{q_0},1-\frac{p}{q},1-\frac{p_1}{q_1}).$$

Lemma 2 gives us that $f\notin \dot M^{p,q}$. If $f_N=\sum_{n<N} 2^{\frac{n(d-\beta)}{p}}\mathbf{1}_{K_n^\beta}$. Lemma 2 gives us moreover that $f_N\in\dot M^{p_1,q_1}$ with $\|f_N\|_{\dot M^{p_1,q_1}}\le C2^{N(d-\beta)(\frac{1}{p}-\frac{1}{p_1})} = C2^{(1-\theta)N(d-\beta)(\frac{1}{p_0}-\frac{1}{p_1})}$, while $f-f_N\in\dot M^{p_0,q_0}$ with $\|f-f_N\|_{\dot M^{p_0,q_0}}\le C2^{N(d-\beta)(\frac{1}{p}-\frac{1}{p_0})} = C2^{-\theta N(d-\beta)(\frac{1}{p_0}-\frac{1}{p_1})}$. Thus $f\in[\dot M^{p_0,q_0},\dot M^{p_1,q_1}]_{\theta,\infty}$. (Remark : this proves that the statement in the introduction of [2] is false). $\diamond$

6 The case $\beta\in\mathbb{N}$.

As we already underlined it, Theorems 2 and 3 are not really new. However, counterexamples were given only for special values of the indexes at stake. Indeed, in [3], [4] or [11], the counterexamples are based on functions $f(x_1,\dots,x_d)=g(x_1,\dots,x_k)$ with $g\in L^p(\mathbb{R}^k)$; such functions f belong to $\mathcal{L}_{p,d-k}(\mathbb{R}^d)=\dot M^{p,\frac{pd}{k}}(\mathbb{R}^d)$. They correspond to integer values of $\beta=d-k$ in our construction.

In order to illustrate those cases, let us consider the function

$$f=\sum_{n\in\mathbb{N}} 2^{n\alpha}\frac{1}{(1+n)^\gamma}\mathbf{1}_{K_n^\beta}$$

for $0<\alpha$ and $0\le\gamma$. When $\beta=d-k$ with $k\in\mathbb{N}^*$, we may choose $K_n^\beta=[0,\frac{1}{2^n}]^k\times[0,1]^{d-k}$. In that case, we find that f is of the same order of magnitude as $\frac{1}{|x'|^\alpha(1+|\ln x'|)^\gamma}\mathbf{1}_{[0,1]^d}$ with $x' = (x_1, \ldots, x_k)$. Thus, we see that our examples are straightforward generalizations of the classical counterexamples.

7 References.

[1] ADAMS, D. R., On the existence of capacitary strong type estimates in $\mathbb{R}^n$, *Ark. Mat.* 14 (1976), 125–140.

[2] BURENKOV, V. I. & NURSULTANOV, E. D., Description of interpolation spaces for local Morrey-type spaces, *Proc. Steklov Inst. Math.* 269 (2010), 46–56.

[3] FEFFERMAN, C. L., The uncertainty principle, *Bull. Amer. Math. Soc.* 9 (1983), 129–206.

[4] KERMAN,R., & SAWYER, E, The trace inequality and eigenvalue estimates for Schrödinger operators, *Ann. Inst. Fourier* 36 (1986), 207–228.

[5] MAZ'YA, V.G., On the theory of the n-dimensional Schrödinger operator, *Izv. Akad. Nauk SSSR*, ser. Mat., 28 (1964), 1145–1172. (in Russian)

[6] MAZ'YA, V.G. & SHAPOSHNIKOVA, T.O., *The theory of multipliers in spaces of differentiable functions*, Pitman, New-York, 1985.

[7] MEYERS, N. G., A theory of capacities for potentials of functions in Lebesgue classes, *Math. Scand.* 26 (1970), 255–292.

[8] PEETRE, J., On the theory of $\mathcal{L}_{p,\lambda}$ spaces, *J. Funct. Anal.* 4 (1969), 71–87.

[11] RUIZ, A. & VEGA, L., Corrigenda to "Unique continuation for Schrödinger operators" and a remark on interpolation of Morrey spaces, *Publ. Mat.* 39 (1995), 404–411.

[9] SAWANO, Y., SUGANO, S. & TANAKA, H., Orlicz–Morrey Spaces and Fractional Operators, *Potential Anal.* 2011, DOI 10.1007/s11118-011-9239-8.

[10] SPANNE, S., Sur l'interpolation entre les espaces $\mathcal{L}_k^{p\Phi}$, *Ann. Scuola Norm. Sup. Pisa* 20 (1966), 625–648.

[12] YANG, Q.X., Characterization of multiplier spaces by wavelets and logarithmic Morrey spaces. Preprint, Wu Han University, 2011.