A remark on the div-curl lemma.

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Abstract : We prove the div-curl lemma for a general class of functional spaces, stable under the action of Calderón–Zygmund operators. The proof is based on a variant of the renormalization of the product introduced by S. Dobyinsky, and on the use of divergence–free wavelet bases.

Introduction.

In 1992, Coifman, Lions, Meyer and Semmes published a paper [COILMS 92] where they gave a new interpretation of the compensated compactness introduced by Murat and Tartar [MUR 78]. They showed that the functions considered by Murat and Tartar had a greater regularity than expected : they belonged to the Hardy space $\mathcal{H}^1$.

They gave a new version of the div-curl lemma of Murat and Tartar :

Theorem 1

If $1 < p < \infty$, $q = p/(p-1)$, $\vec{f} \in (L^p(\mathbb{R}^d))^d$ *and* $\vec{g} \in (L^q)^d$, *then*

$$\mathbf{div}\ \vec{f} = 0 \textit{ and } \mathbf{curl}\ \vec{g} = \vec{0} \ \Rightarrow \vec{f}.\vec{g} \in \mathcal{H}^1$$

There are many proofs of this result. We shall rely mainly on the proof by S. Dobyinsky, based on the renormalization of the product introduced in [DOB 92].

As pointed to me by Prof. Grzegorz Karch, it is easy to see that this result may be extended to a large class of functional spaces. For instance, we have the straightforward consequence of Theorem 1, for the case of weak Lebesgue spaces $L^{p,*}$ (better seen as Lorentz spaces $L^{p,\infty}$) and their preduals $L^{q,1}$:

Corollary 1 :

If $1 < p < \infty$, $q = p/(p-1)$, $\vec{f} \in (L^{p,\infty}(\mathbb{R}^d))^d$ *and* $\vec{g} \in (L^{q,1})^d$, *then*

$$\mathbf{div}\ \vec{f} = 0 \textit{ and } \mathbf{curl}\ \vec{g} = \vec{0} \ \Rightarrow \vec{f}.\vec{g} \in \mathcal{H}^1$$

and

$$\mathbf{div}\ \vec{g} = 0 \textit{ and } \mathbf{curl}\ \vec{f} = \vec{0} \ \Rightarrow \vec{f}.\vec{g} \in \mathcal{H}^1$$

Proof : All we need is the projection operators that lead to the Helmhotz decomposition of a vector field : $Id = \mathbb{P} + \mathbb{Q}$ where $\mathbb{Q}$ is the projection onto irrotational vector fields :

$$\mathbb{Q}\vec{h} = \vec{\nabla}\frac{1}{\Delta}\mathbf{div}\ \vec{h}$$

and $\mathbb{P}$ the projection operator onto solenoidal vector fields. Those projection operators are matrix of singular integral operators and thus are bounded on Lebesque spaces L^r, $1 < r < \infty$,and, by interpolation, on Lorentz spaces spaces $L^{r,t}$, $1 < r < \infty$, $1 \leq t \leq +\infty$.

Let $\epsilon > 0$ such that $\epsilon < \min 1/p, 1/q$. We write $\frac{1}{p_+} = \frac{1}{p} + \epsilon$, $\frac{1}{p_-} = \frac{1}{p} - \epsilon$, $\frac{1}{q_+} = \frac{1}{q} + \epsilon$ and $\frac{1}{q_-} = \frac{1}{q} - \epsilon$. If $\vec{f} \in (L^{p,\infty}(\mathbb{R}^d))^d$, we can write, for every $A > 0$, $\vec{f} = \vec{\alpha}_A + \vec{\beta}_A$ with $\|\vec{\alpha}_A\|_{L^{p_-}} \leq CA\|\vec{f}\|_{L^{p,\infty}}$ and $\|\vec{\beta}_A\|_{L^{p_+}} \leq CA^{-1}\|\vec{f}\|_{L^{p,\infty}}$. If $\mathbf{div}\ \vec{f} = 0$, we have moreover $\vec{f} = \mathbb{P}\vec{f} = \mathbb{P}\vec{\alpha}_A + \mathbb{P}\vec{\beta}_A$. On the other hand, if $\vec{g} \in (L^{q,1})^d$, we can write $\vec{g} = \sum_{j\in\mathbb{N}} \lambda_j \vec{g}_j$ with $\|\vec{g}_j\|_{L^{q_-}}\|\vec{g}\|_{L^{q_+}} \leq 1$ and $\sum_{j\in\mathbb{N}} |\lambda_j| \leq C\|\vec{g}\|_{L^{q,1}}$. If $\mathbf{curl}\ \vec{g} = 0$, we have moreover $\vec{g} = \mathbb{Q}\vec{g} = \sum_{j\in\mathbb{N}} \lambda_j\, \mathbb{Q}\vec{g}_j$. Let $A_j = \|\ \vec{g}_j\|_{L^{q_-}}^{1/2} \|\ \vec{g}_j\|_{L^{q_+}}^{-1/2}$. We then write

$$\vec{f}.\vec{g} = \sum_{j\in\mathbb{N}} \lambda_j\ (\mathbb{P}\vec{\alpha}_{A_j}.\,\mathbb{Q}\vec{g}_j + \mathbb{P}\vec{\beta}_{A_j}.\,\mathbb{Q}\vec{g}_j)$$

and get (from the div-curl theorem of f Coifman, Lions, Meyer and Semmes)

$$\begin{aligned}
\|\vec{f}.\vec{g}\|_{\mathcal{H}^1} &\leq C\sum_{j\in\mathbb{N}} |\lambda_j\ |(\|\mathbb{P}\vec{\alpha}_{A_j}\|_{L^{p_-}}\|\ \mathbb{Q}\vec{g}_j\|_{L^{q_+}} + \|\mathbb{P}\vec{\beta}_{A_j}\|_{L^{p_+}}\|\ \mathbb{Q}\vec{g}_j\|_{L^{q_-}}) \\
&\leq C'\|\vec{f}\|_{L^{p,\infty}} \sum_{j\in\mathbb{N}} |\lambda_j|\ (A_j\|\ \vec{g}_j\|_{L^{q_+}} + {A_j}^{-1}\|\ \vec{g}_j\|_{L^{q_-}}) \\
&= C'\|\vec{f}\|_{L^{p,\infty}} \sum_{j\in\mathbb{N}} |\lambda_j| \\
&\leq C''\|\vec{f}\|_{L^{p,\infty}}\|\vec{g}\|_{L^{q,1}}
\end{aligned}$$

The proof for the case $\mathbf{div}\ \vec{g} = 0$ and $\mathbf{curl}\ \vec{f} = \vec{0}$ is similar. $\diamond$

In this paper, we aim to find a general class of functional spaces for which the div-curl lemma still holds. As we may see from the proof of the Corollary 1, singular integral operators will play a key role in our result. In section 1, we shall introduce Calderòn–Zygmund pairs of functional spaces which will allow us to prove such a general result. In section 2, we recall basics of divergence–free wavelet bases (as described in the book [LEM 02]). In section 3, we prove our main theorem. Then, in section 4, we give examples of Calderòn–Zygmund pairs of functional spaces.

1. Calderón–Zygmund pairs of Banach spaces.

We begin by recalling the definition of a Calderón–Zygmund operator :

Definition 1 :

A) A **singular integral operator** *is a continuous linear mapping from $\mathcal{D}(\mathbb{R}^d)$ to $\mathcal{D}'(\mathbb{R}^d)$ whose distribution kernel $K(x,y) \in \mathcal{D}'(\mathbb{R}^d \times \mathbb{R}^d)$ (defined formally by the formula $Tf(x) = \int K(x,y)f(y)\ dy$) has its restriction outside the diagonal $x = y$ defined by a locally Lipschitz function with the following size estimates :*
i) $\sup_{x\neq y} |K(x,y)||x-y|^d < +\infty$
ii) $\sup_{x\neq y} |\vec{\nabla}_x K(x,y)||x-y|^{d+1} < +\infty$
iii) $\sup_{x\neq y} |\vec{\nabla}_y K(x,y)||x-y|^{d+1} < +\infty$

For such an operater T, we define

$$\|T\|_{SIO} = \| K(x,y)|x-y|^d\|_{L^\infty(\Omega)} + \|\vec{\nabla}_x K(x,y)|x-y|^{d+1}\|_{L^\infty(\Omega)} + \|\vec{\nabla}_y K(x,y)|x-y|^{d+1}\|_{L^\infty(\Omega)}$$

where K is the distribution kernel of T and $\Omega = \mathbb{R}^d \times \mathbb{R}^d - \{(x,y)\ /\ x = y\}$

B) A **Calderón–Zygmund operator** *is a singular integral operator T which may be extended as a bounded operator on L^2 : $\sup_{\varphi\in\mathcal{D},\ \|\varphi\|_2\leq 1} \|T(\varphi)\|_2 < +\infty$.*

We define CZO as the space of Calderón–Zygmund operators, endowed with the norm :

$$\|T\|_{CZO} = \|T\|_{\mathcal{L}(L^2,L^2)} + \|T\|_{SIO}.$$

We may now define our main tool :

Definition 2 :

A **Calderón–Zygmund pair** *of Banach spaces (X,Y) is pair of Banach spaces such that :*
i) we have the continuous embedding : $\mathcal{D}(\mathbb{R}^d) \subset X \subset \mathcal{D}'$ and $\mathcal{D}(\mathbb{R}^d) \subset Y \subset \mathcal{D}'$
iii) Let X_0 be the closure of $\mathcal{D}$ in X; then the dual spaceX_0^* of X_0 (i.e. the space of bounded linear forms on X_0) coincides with Y with equivalence of norms : a distribution T belongs to Y if and anly if there exist a constant C_T such that for all $\varphi \in \mathcal{D}$ we have $|\ \langle T|\varphi\rangle_{\mathcal{D}',\mathcal{D}}\ | \leq C_T\|\varphi\|_X$
iiii) Let Y_0 be the closure of $\mathcal{D}$ in Y; then the dual space Y_0^* of Y_0 coincides with X with equivalence of norms
iv) Every Calderón–Zygmund operator may be extended as a bounded operator on X_0 and on Y_0 : there exists a constant C_0 such that, for every $T \in CZO$ and every $\varphi \in \mathcal{D}$, we have $T(\varphi) \in X_0 \cap Y_0$ and

$$\|T(\varphi)\|_X \leq C_0\|T\|_{CZO}\|\varphi\|_X \text{ and } \|T(\varphi)\|_Y \leq C_0\|T\|_{CZO}\|\varphi\|_Y$$

By duality, we find that every Calderón–Zygmund operator may be extended as a bonded operator on X and Y : if T^* is defined by the formula

$$\langle T(\varphi)|\psi\rangle_{\mathcal{D}',\mathcal{D}} = \langle \varphi|T^*(\psi)\rangle_{\mathcal{D},\mathcal{D}'},$$

then $T \in CZO$ implies $T^* \in CZO$ and we may define $T(f)$ on X as the distribution $\varphi \mapsto \langle f | T^*(\varphi)\rangle_{Y_0^*, Y_0}$. The two definitions of T coincides on X_0.

For $m \in L^\infty$, the operator $T_m : \varphi \mapsto m\varphi$ belongs to CZO (with kernel $K(x, y) = m(x)\delta(x - y)$). The stability of X and Y through multiplication by bounded smooth functions (with the inequalities $\|mf\|_X \leq C_0 \|m\|_\infty \|f\|_X$ and $\|mf\|_Y \leq C_0 \|m\|_\infty \|f\|_Y$) shows that elements of X and Y are (complex) local measures and that X_0 and Y_0 are embedded into L^1_{loc}.

Our main result is then the following one (to be proved in Section 3) :

Theorem 2 :

Let (X, Y) be a Calderón–Zygmund pair of Banach spaces (X, Y). If $\vec{f} \in X_0^d$ and $\vec{g} \in Y^d$, then

$$\mathbf{div}\ \vec{f} = 0 \textit{ and } \mathbf{curl}\ \vec{g} = \vec{0} \ \Rightarrow \vec{f}.\vec{g} \in \mathcal{H}^1$$

and

$$\mathbf{div}\ \vec{g} = 0 \textit{ and } \mathbf{curl}\ \vec{f} = \vec{0} \ \Rightarrow \vec{f}.\vec{g} \in \mathcal{H}^1$$

Remark : The distribution $\vec{f}.\vec{g}$ is well-defined, since $\vec{f} \in X_0^d$: if $\varphi \in \mathcal{D}$, then we have $\varphi\vec{f} \in X_0^d$ and $\vec{g} \in (X_0^*)^d$.

2. Divergence–free wavelet bases.

In this section, we give a short review of properties of divergence-free wavelet bases. Wavelet theory was introduced in the 1980's as an efficient tool for signal analysis. Orthonormal wavelet bases were first constructed by Y. Meyer [LEMM 86], G. Battle [BAT 87] and P.G. Lemarié-Rieusset; a major advance was done with the construction of compactly supported orthonormal wavelets by I. Daubechies [DAU 92]. Then bi-orthogonal bases were introduced by A. Cohen, I. Daubechies and J.C. Feauveau [COHDF 92]. Divergence-free wavelets were introduced by Battle and Federbush [BATF 95]. Compactly divergence-free wavelets were introduced by P.G. Lemarié–Rieusset [LEM 92]; they are not orthogonal wavelets [LEM 94], but have been explored for the numerical analysis of the Navier–Stokes equations [URB 95] [DER 06].

Let $H_{\mathbf{div=0}}$ and $H_{\mathbf{curl=0}}$ be defined as

$$H_{\mathbf{div=0}} = \{\vec{f} \in (L^2)^d \ / \ \mathbf{div}\ \vec{f} = 0\} \text{ and } H_{\mathbf{curl=0}} = \{\vec{f} \in (L^2)^d \ / \ \mathbf{curl}\ \vec{f} = 0\}.$$

For a function $\vec{f} \in (L^2)^d$, $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^d$, we define $\vec{f}_{j,k}$ as $\vec{f}_{j,k}(x) = 2^{jd/2}\vec{f}(2^j x - k)$. Let us recall the mains results of [LEM 92] (described as well in the book [LEM 02]). The idea is to begin with an Hilbertian basis of compactly supported wavelets, associated to a multi-resolution analysis $(V_j)_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$. Associated to this multi-resolution analysis (with orthogonal projection operator Π_j onto V_j), there is a bi-orthogonal multi-resolution analysis (V_j^+) (V_j^-) with projection $\Pi_{(j)}$ onto V_j^- orthogonally to V_j^+ such that $\frac{d}{dx}\circ\Pi_j = \Pi_{(j)}\circ\frac{d}{dx}$.

Starting from this one-dimensional setting, we now consider a bi-orthogonal multi-resolution analysis of $(L^2(\mathbb{R}^d))^d$ $(V_{j,1},\ldots,V_{j,d})$ and $(V_{j,1}^*,\ldots,V_{j,d}^*)$ where $V_{j,k} = V_{j,k,1}\otimes\ldots\otimes V_{j,k,d}$ with $V_{j,k,l} = V_j$ for $k\neq l$ and $V_{j,k,k} = V_j^-$ and $V_{j,k}^* = V_{j,k,1}^*\otimes\ldots\otimes V_{j,k,d}^*$ with $V_{j,k,l}^* = V_j$ for $k\neq l$ and $V_{j,k,k}^* = V_j^+$ Let P_j be the projection operator onto $(V_{j,1},\ldots,V_{j,d})$ orthogonally to $(V_{j,1}^*,\ldots,V_{j,d}^*)$. Its adjoint P_j^* is the projection operator onto $(V_{j,1}^*,\ldots,V_{j,d}^*)$ orthogonally to $(V_{j,1},\ldots,V_{j,d})$. The point is that we have $P_j(\vec{\nabla}f) = \vec{\nabla}(\Pi_j f)$ and $\mathbf{div}\ (P_j^*\vec{f}) = \Pi_j^*(\mathbf{div}\ \vec{f})$.

Those projection operators P_j and P_j^* can give an accurate description of $H_{\mathbf{div}=\mathbf{0}}$ and $H_{\mathbf{curl}=\mathbf{0}}$:

Proposition 1 : (Multi-resolution analysis for divergence-free or irrotational vector fields)

Let $N\in\mathbb{N}$. Then there exists a compact set $K_N\subset\mathbb{R}^d$ such that :

A) **Multi-resolution analysis :** *There exists*

**) functions $\vec{\varphi}_\xi$ and $\vec{\varphi}_\xi^*$ in $(L^2)^d$, $1\le\xi\le d$*

**) functions $\vec{\psi}_\chi$ and $\vec{\psi}_\chi^*$ in $(L^2)^d$, $1\le\chi\le d(2^d-1)$*

such that

i) the functions $\vec{\varphi}_\xi$, $\vec{\varphi}_\xi^*$, $\vec{\psi}_\chi$ and $\vec{\psi}_\chi^*$ are supported in the compact K_N

ii) the functions $\vec{\varphi}_\xi$, $\vec{\varphi}_\xi^*$, $\vec{\psi}_\chi$ and $\vec{\psi}_\chi^*$ are of class $\mathcal{C}^N$

iii) for $l\in\mathbb{N}^d$ with $\sum_{i=1}^d l_i\le N$, we have $\int x^l\vec{\psi}_\chi\ dx = \int x^l\vec{\psi}_\chi^*\ dx = 0$

iv) for j, j', in $\mathbb{Z}$, k, k' in $\mathbb{Z}^d$, ξ, ξ' in $\{1,\ldots,d\}$, and χ, χ' in $\{1,\ldots,d(2^d-1)\}$

$$\int\vec{\varphi}_{\xi,j,k}.\vec{\varphi}_{\xi',j,k'}^*\ dx = \delta_{k,k'}\delta_{\xi,\xi'}\ \text{and}\ \int\vec{\psi}_{\chi,j,k}.\vec{\psi}_{\chi',j',k'}^*\ dx = \delta_{j,j'}\delta_{k,k'}\delta_{\chi,\chi'}$$

v) The projection operators P_j can be defined on $(L^2)^d$ by

$$P_j(\vec{f}) = \sum_{k\in\mathbb{Z}^d}\sum_{1\le\xi\le d}\langle\vec{f}|\vec{\varphi}_{\xi,j,k}^*\rangle\ \vec{\varphi}_{\xi,j,k}.$$

They are bounded on $(L^2)^d$ and satisfy

$$P_j\circ P_{j+1} = P_{j+1}\circ P_j = P_j,\ \lim_{j\to-\infty}\|P_j\vec{f}\|_2 = 0\ \text{and}\ \lim_{j\to+\infty}\|\vec{f}-P_j\vec{f}\|_2 = 0.$$

vi) The operators Q_j defined on $(L^2)^d$ by

$$Q_j(\vec{f}) = \sum_{k\in\mathbb{Z}^d}\sum_{1\le\chi\le d(2^d-1)}\langle\vec{f}|\vec{\psi}_{\chi,j,k}^*\rangle\ \vec{\psi}_{\chi,j,k}$$

are bounded on $(L^2)^d$ and satisfy

$$Q_j = P_{j+1} - P_j$$

and

$$\|\vec{f}\|_2 \approx \sqrt{\sum_{j\in\mathbb{Z}} \|Q_j\vec{f}\|_2^2} \approx \sqrt{\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}\sum_{1\le\chi\le d(2^d-1)} |\ \langle\vec{f}|\vec{\psi}^*_{\chi,j,k}\rangle\ |^2}$$

B) **Irrotational vector fields :** *The projection operators P_j satisfy :*

$$\vec{f} \in (L^2)^d \text{ and } \mathbf{curl}\ \vec{f} = 0 \Rightarrow \mathbf{curl}\ P_j(\vec{f}) = 0$$

Moreover, there exists
) $2^d - 1$ functions $\vec{\gamma}_\eta \in (L^2)^d$, $1 \le \eta \le 2^d -)$ with* **curl $\vec{\gamma}_\eta = 0$
**) $2^d - 1$ functions $\vec{\gamma}^*_\eta \in (L^2)^d$, $1 \le \eta \le 2^d - 1$*
such that
i) the functions $\vec{\gamma}_\eta$ and $\vec{\gamma}^*_\eta$ are supported in the compact K_N
ii) the functions $\vec{\gamma}_\eta$ and $\vec{\gamma}^*_\eta$ are of class $\mathcal{C}^N$
iii) for $l \in \mathbb{N}^d$ with $\sum_{i=1}^d l_i \le N$, we have $\int x^l\vec{\gamma}_\eta\ dx = \int x^l\vec{\gamma}^*_\eta\ dx = 0$
iv) for j, j', in $\mathbb{Z}$, k, k' in $\mathbb{Z}^d$ and η, η' in $\{1,\ldots,2^d-1\}$,

$$\int \vec{\gamma}_{\eta,j,k}.\vec{\gamma}^*_{\eta',j',k'}\ dx = \delta_{j,j'}\delta_{k,k'}\delta_{\eta,\eta'}$$

vi) The operators S_j defined on $(L^2)^d$ by

$$S_j(\vec{f}) = \sum_{k\in\mathbb{Z}^d}\sum_{1\le\eta\le 2^d-1} \langle\vec{f}|\vec{\gamma}^*_{\eta,j,k}\rangle\ \vec{\gamma}_{\eta,j,k}$$

are bounded on $(L^2)^d$ and satisfy

$$\forall\vec{f} \in H_{\mathbf{curl=0}} \quad S_j\vec{f} = Q_j\vec{f}$$

and

$$\forall\vec{f} \in H_{\mathbf{curl=0}} \quad \|\vec{f}\|_2\ \approx \sqrt{\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}\sum_{1\le\eta\le 2^d-1} |\ \langle\vec{f}|\vec{\gamma}^*_{\eta,j,k}\rangle\ |^2}$$

C) **Divergence-free vector fields :** *The projection operators P_j satisfy :*

$$\vec{f} \in (L^2)^d \text{ and } \mathbf{div}\ \vec{f} = 0 \Rightarrow \mathbf{div}\ P^*_j(\vec{f}) = 0$$

Moreover, there exists
) $(d-1)(2^d-1)$ functions $\vec{\alpha}_\epsilon \in (L^2)^d$, $1 \le \epsilon \le (d-1)(2^d-1)$ with* **div $\vec{\alpha}_\epsilon = 0$
**) $(d-1)(2^d-1)$ functions $\vec{\alpha}^*_\epsilon \in (L^2)^d$, $1 \le \epsilon \le (d-1)(2^d-1)$*

such that
i) the functions $\vec{\alpha}_\epsilon$ *and* $\vec{\alpha}^*_\epsilon$ *are supported in the compact* K_N
ii) the functions $\vec{\alpha}_\epsilon$ *and* $\vec{\alpha}^*_\epsilon$ *are of class* $\mathcal{C}^N$
iii) for $l \in \mathbb{N}^d$ *with* $\sum_{i=1}^d l_i \leq N$*, we have* $\int x^l \vec{\alpha}_\epsilon \, dx = \int x^l \vec{\alpha}^*_\epsilon \, dx = 0$
iv) for j*,* j'*, in* $\mathbb{Z}$*,* k*,* k' *in* $\mathbb{Z}^d$ *and* ϵ*,* ϵ' *in* $\{1, \ldots, (d-1)(2^d-1)\}$*,*

$$\int \vec{\alpha}_{\epsilon,j,k}.\vec{\alpha}^*_{\epsilon',j',k'} \, dx = \delta_{j,j'}\delta_{k,k'}\delta_{\epsilon,\epsilon'}$$

vi) The operators R_j *defined on* $(L^2)^d$ *by*

$$R_j(\vec{f}) = \sum_{k \in \mathbb{Z}^d} \sum_{1 \leq \epsilon \leq (d-1)(2^d-1)} \langle \vec{f} | \vec{\alpha}^*_{\epsilon,j,k} \rangle \; \vec{\alpha}_{\epsilon,j,k}$$

are bounded on $(L^2)^d$ *and satisfy*

$$\forall \vec{f} \in H_{\mathbf{div=0}} \quad R_j \vec{f} = Q_j^* \vec{f}$$

and

$$\forall \vec{f} \in H_{\mathbf{div=0}} \quad \|\vec{f}\|_2 \; \approx \sqrt{\sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \sum_{1 \leq \epsilon \leq (d-1)(2^d-1)} | \, \langle \vec{f} | \vec{\alpha}^*_{\epsilon,j,k} \rangle \, |^2}$$

We would like now to use those special functions with our spaces X and Y. We begin with the following lemma :

Lemma 1 :
a) If $\vec{f} \in X_0^d$*, then* $P_j \vec{f}$ *and* $P_j^* \vec{f}$ *converge strongly to* 0 *in* X^d *as* $j \to -\infty$ *and converge strongly to* $\vec{f}$ *in* X^d *as* $j \to +\infty$*.*

b) If $\vec{f} \in Y_0^d$*, then* $P_j \vec{f}$ *and* $P_j^* \vec{f}$ *converge strongly to* 0 *in* Y^d *as* $j \to -\infty$ *and converge strongly to* $\vec{f}$ *in* Y^d *as* $j \to +\infty$*.*

c) If $\vec{f} \in X^d$*, then* $P_j \vec{f}$ *and* $P_j^* \vec{f}$ *converge *-weakly to* 0 *in* X^d *as* $j \to -\infty$ *and converge *-weakly to* $\vec{f}$ *in* X^d *as* $j \to +\infty$*.*

d) If $\vec{f} \in Y^d$*, then* $P_j \vec{f}$ *and* $P_j^* \vec{f}$ *converge *-weakly to* 0 *in* Y^d *as* $j \to -\infty$ *and converge *-weakly to* $\vec{f}$ *in* Y^d *as* $j \to +\infty$*.*

Proof : First, we check that the operators are well defined. If $f \in \mathcal{C}^N$ has a compact support, then we may write $f = f\theta$, with $\theta \in \mathcal{D}$ equal to 1 on a neighborhood of the support of f. Thus, $f = T_f(\theta)$ and we find that $f \in X_0 \cap Y_0$. Thus, $\langle f|g \rangle_{X_0,Y}$ is well defined for every $g \in Y$, and $\langle f|h\rangle_{Y_0,X}$ is well defined for every $h \in X$. We may then consider the operators on X^d :

$$P_j(\vec{f}) = \sum_{k\in\mathbb{Z}^d} \sum_{1\leq\xi\leq d} \langle \vec{f}|\vec{\varphi}^*_{\xi,j,k}\rangle_{X,Y_0}\ \vec{\varphi}_{\xi,j,k}$$

and

$$P^*_j(\vec{f}) = \sum_{k\in\mathbb{Z}^d} \sum_{1\leq\xi\leq d} \langle \vec{f}|\vec{\varphi}_{\xi,j,k}\rangle_{X,Y_0}\ \vec{\varphi}^*_{\xi,j,k}.$$

We have $\sup_{j\in\mathbb{Z}} \|P_j\|_{CZO} = \sup_{j\in\mathbb{Z}} \|P^*_j\|_{CZO} < +\infty$. Thus, those operators are equicontinuous on X^d.

To prove a), we need to check the limits only on a dense subspace of X_0^d. X_0 cannot be embedded into L^1 : if $f \in \mathcal{D}$ with $\hat{f}(0) \neq 0$, then the Riesz transforms $R_j f$ are not in L^1 but belong to X_0. It means that $f \in \mathcal{D} \mapsto \|f\|_1$ is not continuous for the X_0 norm. We may find a sequence of functions f_n such that $\|f_n\|_X$ converge to 0 and $\|f_n\|_1 = 1$. Since $|f_n|$ is Lipschitz and compactly supported, we can regularize f_n and find a sequence of smooth compactly supported functions $f_{n,k}$ such that all the $f_{n,k}$, $k \in \mathbb{N}$, are supported in a compact neighborhood of the support of f_n and converge, as $k \to +\infty$, uniformly to $|f_n|$; then, we have convergence in X (since $Y_0 \subset L^1_{\rm loc}$) and in L^1. Thus, we can find a sequence of functions f_n which are in $\mathcal{D}$, with $\int f_n\, dx = 1$ and $\lim_{n\to+\infty} \|f_n\|_X = 0$. This gives that the set of function $f \in \mathcal{D}$ with $\int f\, dx = 0$ is dense in X_0.

We now consider $Q_j = P_{j+1} - P_j$ and $Q^*_j = P^*_{j+1} - P^*_j$:

$$Q_j(\vec{f}) = \sum_{k\in\mathbb{Z}^d} \sum_{1\leq\chi\leq d(2^d-1)} \langle \vec{f}|\vec{\psi}^*_{\chi,j,k}\rangle_{X,Y_0}\ \vec{\psi}_{\chi,j,k}$$

and

$$Q^*_j(\vec{f}) = \sum_{k\in\mathbb{Z}^d} \sum_{1\leq\chi\leq d(2^d-1)} \langle \vec{f}|\vec{\psi}_{\chi,j,k}\rangle_{X,Y_0}\ \vec{\psi}^*_{\chi,j,k}.$$

If $\vec{f} \in \mathcal{D}^d$ and $\int \vec{f}\, dx = 0$, we have $\vec{f} = \sum_{1\leq l\leq d} \partial_l \vec{f}_l$ for some $\vec{f}_l \in \mathcal{D}^d$. Similarly, we have $\psi^*_\chi = \sum_{1\leq l\leq d} \partial_l \vec{\Psi}^*_{\chi,l}$ and $\psi_\chi = \sum_{1\leq l\leq d} \partial_l \vec{\Psi}_{\chi,l}$ for some compactly supported functions of class $\mathcal{C}^N$. Thus, we find that, for $\vec{f} \in \mathcal{D}^d$ with $\int \vec{f}\, dx = 0$,

$$\|P_{j+1}\vec{f} - P_j\vec{f}\|_X + \|P^*_{j+1}\vec{f} - P^*_j\vec{f}\|_X \leq C \min(\sum_{l=1^d} \|\partial_l\vec{f}\|_X 2^j, \sum_{i=1}^{d} \|\vec{f}_l\|_X 2^{-j}).$$

Thus, $P_j\vec{f}$ and $P^*_j\vec{f}$ have strong limits in X_0^d when j goes to $-\infty$ or $+\infty$. If $\vec{g} \in \mathcal{D}^d$, we write $\vec{f} \in (L^2)^d$ and $\vec{g} \in (L^2)^d$, and see that

$$\lim_{j\to-\infty} \langle P_j\vec{f}|\vec{g}\rangle_{X,Y_0} = \lim_{j\to-\infty} \langle P^*_j\vec{f}|\vec{g}\rangle_{X,Y_0} = 0$$

and

$$\lim_{j\to+\infty}\langle P_j\vec f|\vec g\rangle_{X,Y_0}=\lim_{j\to+\infty}\langle P_j^*\vec f|\vec g\rangle_{X,Y_0}=\langle\vec f|\vec g\rangle_{X,Y_0}.$$

Thus, we have, for $\vec f\in\mathcal{D}^d$ with $\int\vec f\,dx=0$,

$$\lim_{j\to-\infty}\|P_j\vec f\|_X=\lim_{j\to-\infty}\|P_j^*\vec f\|_X=0$$

and

$$\lim_{j\to+\infty}\|P_j\vec f-\vec f\|_X=\lim_{j\to+\infty}\|P_j^*\vec f-\vec f\|_X=0$$

Thus a) is proved. b) is proved in a similar way. By duality, we get c) and d). $\diamond$

We may now consider the operators :

$$R_j(\vec f)=\sum_{k\in\mathbb{Z}^d}\sum_{1\le\epsilon\le(d-1)(2^d-1)}\langle\vec f|\vec\alpha^*_{\epsilon,j,k}\rangle_{X,Y_0}\ \vec\alpha_{\epsilon,j,k}$$

and

$$S_j(\vec f)=\sum_{k\in\mathbb{Z}^d}\sum_{1\le\eta\le 2^d-1}\langle\vec f|\vec\gamma^*_{\eta,j,k}\rangle_{X,Y_0}\ \vec\gamma_{\eta,j,k}.$$

From the identities $\mathbb{P}^*R_j^*=\mathbb{P}^*Q_j$ and $\mathbb{Q}^*S_j^*=\mathbb{Q}^*Q_j^*$ which are valid from $\mathcal{D}^d$ to Y_0^d, we find by duality that $R_j\mathbb{P}=Q_j^*\mathbb{P}$ and $S_j\,\mathbb{Q}=Q_j\,\mathbb{Q}$ on X^d. To be able to use those identities, we shall need the following lemma :

Lemma 2 :
Let $\vec f\in X^d$. Then :
i) $\mathbb{P}\vec f=\vec f\Leftrightarrow\text{div }\vec f=0$
ii) $\mathbb{Q}\vec f=\vec f\Leftrightarrow\text{curl }\vec f=0$

Proof : First, we check that $f\in X$ and $\Delta f=0\Rightarrow f=0$. Take $\theta\in\mathcal{D}$ such that $\theta\ge 0$, and $\theta\ne 0$, and define $\gamma=\frac{1}{(1+x^2)^{\frac{n+1}{2}}}*\theta$. Convolution with the kernel $\frac{1}{(1+x^2)^{\frac{n+1}{2}}}$ is a Calderón–Zygmund operator, so we get that $\gamma\in Y_0$. Moreover, if g is a function such that $(1+x^2)^{\frac{n+1}{2}}g\in L^\infty$, we find that $g=\gamma^{-1}g\gamma=T_{\gamma^{-1}g}(\gamma)$, where the pointwise multiplication operator $T_{\gamma^{-1}g}$ is a Calderón–Zygmund operator, so we get that $g\in Y_0$. This proves that $X\subset\mathcal{S}'$. Thus, if $f\in X$ and $\Delta f=0$, we find that f is a harmonic polynomial. Moreover $\int|f|\gamma\,dx=\langle f|T_{\frac{f}{|f|}}(\gamma)\rangle_{X,Y_0}$, hence the integral $\int|f|\gamma\,dx$ must be finite, and f must be constant. As the smooth functions with vanishing integral are dense in Y_0, we find that the constant is equal to 0.

Now, we have for a distribution $\vec f$ that

$$\text{div }\vec f=0\Leftrightarrow\forall\vec\varphi\in\mathcal{D}^d\text{ with curl }\vec\varphi=0,\quad\langle\vec f|\vec\varphi\rangle=0;$$

thus, we have on X^d that div $\mathbb{P}\vec{f}=0$. Similarly, we have for a distribution $\vec{f}$ that

$$\text{curl } \vec{f}=0 \Leftrightarrow \forall \vec{\varphi} \in \mathcal{D}^d \text{ with div } \vec{\varphi}=0, \quad \langle \vec{f} | \vec{\varphi} \rangle = 0;$$

thus, we have on X^d that curl $\mathbb{Q}\vec{f}=0$.

Conversely, we start from the decomposition $Id = \mathbb{P} + \mathbb{Q}$ valid on X^d. If div $\vec{f}=0$, then we find that $\vec{h}=\vec{f}-\mathbb{P}\vec{f}=\mathbb{Q}\vec{f}$ satisfies. div $\vec{h}=0$ and curl $\vec{h}=0$. But this implies that $\Delta\vec{h}=0$, hence $\vec{h}=0$ We prove similarly that curl $\vec{f}=0$ implies that $f=\mathbb{Q}\vec{f}$. $\diamond$

3. The proof of the div-curl lemma.

As in [LEM 02], we prove Theorem 2 by adapting the proof given by Dobyinsky [DOB 92]. This proof uses the renormalization of the product through wavelet bases.

If $\vec{f} \in X^d$, $\vec{g} \in Y^d$ and if moreover $\vec{f} \in X_0^d$ or $\vec{g} \in Y_0^d$, we use lemma 1 to get that, in the distribution sense, we have

$$\vec{f}.\vec{g} = \lim_{j \to +\infty} P_j^*.\vec{f}.P_j\vec{g} - P_{-j}^*.\vec{f}.P_{-j}\vec{g}$$

and thus

$$\vec{f}.\vec{g} = \sum_{j \in \mathbb{Z}} P_j^*.\vec{f}.Q_j\vec{g} + Q_j^*\vec{f}.P_j\vec{g} + Q_j\vec{f}.Q_j^*\vec{g}.$$

If moreover div $\vec{f}=0$ and curl $\vec{g}=0$, we use lemma 2 to get that

$$\vec{f}.\vec{g} = \sum_{j \in \mathbb{Z}} P_j^*\vec{f}.S_j\vec{g} + R_j\vec{f}.P_j\vec{g} + R_j\vec{f}.S_j\vec{g}.$$

We shall prove that the three terms

$$A(\vec{f},\vec{g}) = \sum_{j \in \mathbb{Z}} P_j^*\vec{f}.S_j\vec{g},$$

$$B(\vec{f},\vec{g}) = \sum_{j \in \mathbb{Z}} R_j\vec{f}.P_j\vec{g}$$

and

$$C(\vec{f},\vec{g}) = \sum_{j \in \mathbb{Z}} R_j\vec{f}.S_j\vec{g}$$

belong to $\mathcal{H}^1$.

We make the proof in the case $\vec{f} \in X_0^d$ (the proof is similar in the case $\vec{g} \in Y_0^d$). We first check that A and B map $(X_0)^d \times Y^d$ to $\mathcal{H}^1$: we use the duality of H^1 and CMO (the closure of $\mathcal{C}_0$ in BMO) (see Coifman and Weiss [COIW 77] and Bourdaud [BOU 02]) and try to prove that the operators

$$\mathcal{A}(\vec{f}, h) = \sum_{j\in\mathbb{Z}} S_j^*(hP_j^*\vec{f})$$

and

$$\mathcal{B}(\vec{f}, h) = \sum_{j\in\mathbb{Z}} P_j^*(hR_j\vec{f})$$

map $(X_0)^d \times CMO$ to $(X_0)^d$.

In order to prove this, we shall prove that $\mathcal{A}(., h)$ and that $\mathcal{B}(.h)$ are matrices of singular integral operators when $h \in \mathcal{D}$ and that we have the estimates $\|\mathcal{A}(., h)\|_{CZO} \leq C\|h\|_{BMO}$ and $\|\mathcal{B}(., h)\|_{CZO} \leq C\|h\|_{BMO}$. For $\mathcal{B}$, we may as well study the adjoint operator

$$\mathcal{B}^*(\vec{f}, h) = \sum_{j\in\mathbb{Z}} R_j^*(hP_j\vec{f})$$

First, we estimate the size of the kernels and of their gradients. The kernels $A_h(x, y)$ of $\mathcal{A}(., h)$ and $B_h^*(x, y)$ of $\mathcal{B}(., h)^*$ are given by

$$A_h(x, y) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} \sum_{1\leq\xi\leq d} \sum_{l\in\mathbb{Z}^d} \sum_{1\leq\eta\leq 2^d-1} \vec{\gamma}^*_{\eta,j,l}(x)\langle h\vec{\varphi}^*_{\xi,j,k}|\vec{\gamma}_{\eta,j,l}\rangle\vec{\varphi}_{\xi,j,k}(y)$$

and

$$B_h^*(x, y) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} \sum_{1\leq\xi\leq d} \sum_{l\in\mathbb{Z}^d} \sum_{1\leq\epsilon\leq(d-1)(2^d-1)} \vec{\alpha}^*_{\epsilon,j,l}(x)\langle h\vec{\varphi}_{\xi,j,l}|\vec{\alpha}_{\epsilon,j,k}\rangle\vec{\varphi}^*_{\xi,j,k}(y)$$

There are only a few terms that interact, because of the localization of the supports : if $K_N \subset B(0, M)$, then $\langle h\vec{\varphi}^*_{\xi,j,k}|\vec{\gamma}_{\eta,j,l}\rangle = \langle h\vec{\varphi}_{\xi,j,l}|\vec{\alpha}_{\epsilon,j,k}\rangle = 0$ if $|l - k| > 2M$. Let

$$C(h) = \sup_{j\in\mathbb{Z},k\in\mathbb{Z}^d,1\leq\xi\leq d,l\in\mathbb{Z}^d,1\leq\eta\leq 2^d-1} |\ \langle h\vec{\varphi}^*_{\xi,j,k}|\vec{\gamma}_{\eta,j,l}\rangle|$$

and

$$D(h) = \sup_{j\in\mathbb{Z},k\in\mathbb{Z}^d,1\leq\xi\leq d,l\in\mathbb{Z}^d,1\leq\epsilon\leq(d-1)(2^d-1)} |\ \langle h\vec{\varphi}_{\xi,j,l}|\vec{\alpha}_{\epsilon,j,k}\rangle|$$

Then we have

$$|A_h(x, y)| \leq \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} CC(h)2^{jd}\mathbf{1}_{B(0,M)}(2^jx - k)\mathbf{1}_{B(0,3M)}(2^jy - k)$$

and thus

$$|A_h(x, y)| \leq CC(h) \sum_{2^j|y-x|\leq 4M} 2^{jd} \leq C'C(h)|x - y|^{-d}$$

and similarly
$$|B_h(x,y)| \leq CD(h)|x-y|^{-d}.$$
In the same way, we have
$$|\vec{\nabla}_x A_h(x,y)| + |\vec{\nabla}_y A_h(x,y)| \leq \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} CC(h)2^{j(d+1)}\mathbf{1}_{B(0,M)}(2^jx-k)\mathbf{1}_{B(0,3M)}(2^jy-k)$$
and thus
$$|\vec{\nabla}_x A_h(x,y)| + |\vec{\nabla}_y A_h(x,y)| \leq CC(h)|x-y|^{-d-1}$$
and similarly
$$|\vec{\nabla}_x B_h(x,y)| + |\vec{\nabla}_y B_h(x,y)| \leq CD(h)|x-y|^{-d-1}.$$
Moreover, the function $\vec{\varphi}^*_{\xi,j,k}.\vec{\gamma}_{\eta,j,l}$ is supported in $B(2^{-j}k, M2^{-j})$, $\|\vec{\varphi}^*_{\xi,j,k}.\vec{\gamma}_{\eta,j,l}\|_\infty \leq C2^{jd}$ and $\int \vec{\varphi}^*_{\xi,j,k}.\vec{\gamma}_{\eta,j,l}\ dx = 0$ (since $P_j^*\vec{\varphi}^*_{\xi,j,k}. = \vec{\varphi}^*_{\xi,j,k}.$ and $Q_j\vec{\gamma}_{\eta,j,l} = \vec{\gamma}_{\eta,j,l}$). Thus, we find that $\|\vec{\varphi}^*_{\xi,j,k}.\vec{\gamma}_{\eta,j,l}\|_{\mathcal{H}^1} \leq C$, so that
$$C(h) \leq C\|h\|_{BMO}.$$
We have similar estimates for $\|\vec{\varphi}_{\xi,j,l}.\vec{\alpha}_{\epsilon,j,k}\|_{\mathcal{H}^1}$ (since $P_j\vec{\varphi}_{\xi,j,l}. = \vec{\varphi}_{\xi,j,l}.$ and $Q_j^*\vec{\alpha}_{\epsilon,j,k} = \vec{\alpha}_{\epsilon,j,k}$, and thus $\int \vec{\varphi}_{\xi,j,l}.\vec{\alpha}_{\epsilon,j,k}\ dx = 0$), and thus
$$D(h) \leq C\|h\|_{BMO}.$$

Thus far, we have proven that $\mathcal{A}(.,h)$ and $\mathcal{B}(.,h)$ are singular integral operators. To prove L^2 boundedness, we use the $T(1)$ theorem of David and Journé [DAVJ 84]. We've got to check that the operators are weakly bounded (in the sense of the WBP property), and to compute the images of the function $f=1$ through the operators and through their adjoints.

Let $x_0 \in \mathbb{R}^d$, $r_0 > 0$ and let $\vec{f}$ and $\vec{g}$ be supported in $B(x_0,r_0)$. We want to estimate $\langle \mathcal{A}(\vec{f},h)|\vec{g}\rangle_{\mathcal{D}',\mathcal{D}}$ and $\langle \mathcal{B}(\vec{f},h)|\vec{g}\rangle_{\mathcal{D}',\mathcal{D}}$. We have $|\langle \mathcal{A}(\vec{f},h)|\vec{g}\rangle_{\mathcal{D}',\mathcal{D}}| \leq \sum_{j\in\mathbb{Z}} A_j$ where
$$A_j = \sum_{k\in\mathbb{Z}^d} \sum_{1\leq\xi\leq d} \sum_{l\in\mathbb{Z}^d} \sum_{1\leq\eta\leq 2^d-1} \left| \ \langle \vec{g}|\vec{\gamma}^*_{\eta,j,l}\rangle\ \langle h\vec{\varphi}^*_{\xi,j,k}|\vec{\gamma}_{\eta,j,l}\rangle\ \langle \vec{f}|\vec{\varphi}_{\xi,j,k}\rangle\ \right|$$
and similarly $|\langle \mathcal{B}(\vec{f},h)|\vec{g}\rangle_{\mathcal{D}',\mathcal{D}} \leq \sum_{j\in\mathbb{Z}} B_j$ where
$$B_j = \sum_{k\in\mathbb{Z}^d} \sum_{1\leq\xi\leq d} \sum_{l\in\mathbb{Z}^d} \sum_{1\leq\epsilon\leq(d-1)(2^d-1)} \left| \ \langle \vec{g}|\vec{\varphi}_{\xi,j,l}\rangle\ \langle h\vec{\varphi}_{\xi,j,l}|\vec{\alpha}_{\epsilon,j,k}\rangle\ \langle \vec{f}|\vec{\alpha}^*_{\epsilon,j,k}\rangle\ \right|$$
We have
$$A_j \leq C(h) \sum_{k\in\mathbb{Z}^d} \sum_{1\leq\xi\leq d} \sum_{|l-k|\leq 2M} \sum_{1\leq\eta\leq 2^d-1} \left| \ \langle \vec{g}|\vec{\gamma}^*_{\eta,j,l}\rangle\ \langle \vec{f}|\vec{\varphi}_{\xi,j,k}\rangle\ \right|$$

which gives

$$A_j \leq C(h) \sum_{k\in\mathbb{Z}^d} \sum_{1\leq\xi\leq d} \mid \langle \vec{f} | \vec{\varphi}_{\xi,j,k} \rangle \mid \sum_{l\in\mathbb{Z}^d} \sum_{1\leq\eta\leq 2^d-1} \mid \langle \vec{g} | \vec{\gamma}^*_{\eta,j,l} \rangle \mid \leq CC(h) 2^{jd} \|\vec{f}\|_1 \|\vec{g}\|_1$$

and

$$A_j \leq C\ C(h) \sqrt{\sum_{k\in\mathbb{Z}^d} \sum_{1\leq\xi\leq d} \mid \langle \vec{f} | \vec{\varphi}_{\xi,j,k} \rangle \mid^2} \sqrt{\sum_{l\in\mathbb{Z}^d} \sum_{1\leq\eta\leq 2^d-1} \mid \langle \vec{g} | \vec{\gamma}^*_{\eta,j,l} \rangle \mid^2}$$

and thus

$$A_j \leq C' C(h) \|S_j \vec{g}\|_2 \|P_j^* \vec{f}\|_2 \leq C'' C(h) 2^{-j} \|\vec{\nabla} \vec{g}\|_2 \|\vec{f}\|_2.$$

Finally, we get

$$\begin{array}{rcl} \mid \langle \mathcal{A}(\vec{f}, h) | \vec{g} \rangle_{\mathcal{D}',\mathcal{D}} | & \leq & CC(h) (\sum_{2^j r_0 \leq 1} 2^{jd} r_0^d \|\vec{f}\|_2 \|\vec{g}\|_2 + \sum_{2^j r_0 > 1} 2^{-j} \|\vec{\nabla} \vec{g}\|_2 \|\vec{f}\|_2) \\ & \leq & C' C(h) (\|\vec{f}\|_2 + r_0 \|\vec{\nabla} \vec{f}\|_2)(\|\vec{g}\|_2 + r_0 \|\vec{\nabla} g\|_2). \end{array}$$

Similar computations (based on the inequality $\|R_j(\vec{f})\|_2 \leq C 2^{-j} \|\vec{\nabla} \vec{f}\|_2$) gives as well

$$|\langle \mathcal{B}(\vec{f}, h) | \vec{g} \rangle_{\mathcal{D}',\mathcal{D}}| \leq CD(h)(\|\vec{f}\|_2 + r_0 \|\vec{\nabla} \vec{f}\|_2)(\|\vec{g}\|_2 + r_0 \|\vec{\nabla} g\|_2).$$

Thus, our operators satisfy the weak boundedness property.

We must now compute the distributions $T(1)$ and $T^*(1)$ when T is one component of the matrix of operators $\mathcal{A}(., h)$ or of $\mathcal{B}(., h)$. We must prove that, if $\theta \in \mathcal{D}$ is equal to 1 on a neighborhood of 0, if $\vec{\theta}_{l,R} = (\theta_{1,l,R}, \ldots, \theta_{d,l,R})$ with $\theta_{k,l,R} = \delta_{k,l} \theta(\frac{x}{R})$ and if $\vec{\psi} \in \mathcal{D}^d$ with $\int \psi\ dx = 0$, then we have

$$\lim_{R\to+\infty} \sum_{j\in\mathbb{Z}} S_j^*(h P_j^* \vec{\theta}_{l,R}) \in (BMO)^d$$

(the limit is taken in $(\mathcal{D}'/\mathbb{R})^d$) and similarly that

$$\lim_{R\to+\infty} \sum_{j\in\mathbb{Z}} P_j(h S_j \vec{\theta}_{l,R}) \in (BMO)^d$$

$$\lim_{R\to+\infty} \sum_{j\in\mathbb{Z}} P_j^*(h R_j \vec{\theta}_{l,R}) \in (BMO)^d$$

and

$$\lim_{R\to+\infty} \sum_{j\in\mathbb{Z}} R_j^*(h P_j \vec{\theta}_{l,R}) \in (BMO)^d$$

To check that, we write $\vec{h}_l = (h_{1,l}, \ldots, h_{d,l})$ with $h_{k,l} = \delta_{k,l} h$ and we consider $\vec{\psi} \in \mathcal{D}^d$ with $\int \psi\ dx = 0$. We have $\sum_{j\in\mathbb{Z}} \|S_j(\vec{\psi})\|_1 < +\infty$ and $\|h P_j^* \vec{\theta}_{l,R}\|_\infty \leq \|h\|_\infty \|\theta\|_\infty$ and thus we get by dominated convergence that

$$\lim_{R\to+\infty}\int \vec\psi.\sum_{j\in\mathbb{Z}} S_j^*(hP_j^*\vec\theta_{l,R})\ dx = \sum_{j\in\mathbb{Z}}\int S_j\vec\psi.\vec h_l\ dx.$$

$\sum_{j\in\mathbb{Z}} S_j$ is a matrix of Calderón–Zygmund operators T which satisfy $T^*(1)=0$, hence map $\mathcal{H}^1$ to $\mathcal{H}^1$, so that we find

$$|\sum_{j\in\mathbb{Z}}\int S_j\vec\psi.\vec h_l\ dx| \le C\|h\|_{BMO}\|\vec\psi\|_{\mathcal{H}^1}$$

and thus $\lim_{R\to+\infty}\sum_{j\in\mathbb{Z}} S_j^*(hP_j^*\vec\theta_{l,R})\in(BMO)^d$. Similar estimates prove that

$$\lim_{R\to+\infty}\int \vec\psi.R_j^*(hP_j\vec\theta_{l,R})\ dx = \sum_{j\in\mathbb{Z}}\int R_j\vec\psi.\vec h_l\ dx.$$

and

$$|\sum_{j\in\mathbb{Z}}\int R_j\vec\psi.\vec h_l\ dx| \le C\|h\|_{BMO}\|\vec\psi\|_{\mathcal{H}^1}$$

so that $\lim_{R\to+\infty}\sum_{j\in\mathbb{Z}} R_j^*(hP_j\vec\theta_{l,R})\in(BMO)^d$.

On the other hand, we have

$$\begin{aligned}|\textstyle\int \vec\psi.P_j(hS_j\vec\theta_{l,R})\ dx| &\le C\|h\|_\infty\|P_j\vec\psi\|_1\|S_j\vec\theta_{l,R}\|_\infty\\ &\le C_{\vec\psi}\|h\|_\infty\min(1,2^j)\min(\|\theta\|_\infty,2^{-j}R^{-1}\|\vec\nabla\theta\|_\infty)=O(R^{-1/2})\end{aligned}$$

so that $\lim_{R\to+\infty}\sum_{j\in\mathbb{Z}} P_j(hS_j\vec\theta_{l,R})=0$. Similarly, we have $\lim_{R\to+\infty}\sum_{j\in\mathbb{Z}} P_j^*(hR_j\vec\theta_{l,R}) = 0$.

Thus, we have proved that $\mathcal{A}$ and $\mathcal{B}$ map $X_0^d\times CMO$ to X_0^d, and thus that A and B map $X_0^d\times Y^d$ to $\mathcal{H}^1$. We still have to deal with $C(\vec f,\vec g)=\sum_{j\in\mathbb{Z}} R_j\vec f.S_j\vec g$. We write

$$C(\vec f,\vec g)=\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}\sum_{1\le\eta\le 2^d-1}\sum_{l\in\mathbb{Z}^d}\sum_{1\le\epsilon\le(d-1)(2^d-1)}\langle\vec g|\vec\gamma^*_{\eta,j,k}\rangle\ \langle\vec f|\vec\alpha^*_{\epsilon,j,l}\rangle\ \vec\alpha_{\epsilon,j,l}.\vec\gamma_{\eta,j,k}$$

We have $\vec\alpha_{\epsilon,j,l}.\vec\gamma_{\eta,j,k}=0$ for $|k-l|>2M$ and $\|\vec\alpha_{\epsilon,j,l}.\vec\gamma_{\eta,j,k}\|_{\mathcal{H}^1}\le C$ for $|k-l|\le 2M$. Thus, we are lead to prove that :

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}\sum_{1\le\eta\le 2^d-1}\sum_{|l-k|\le 2M}\sum_{1\le\epsilon\le(d-1)(2^d-1)}|\ \langle\vec g|\vec\gamma^*_{\eta,j,k}\rangle|\ |\langle\vec f|\vec\alpha^*_{\epsilon,j,l}\rangle|\le C\|\vec f\|_{X_0^d}\|\vec g\|_{Y^d}.$$

For $1\le\eta\le 2^d-1$, $1\le\epsilon\le(d-1)(2^d-1)$ and $r\in\mathbb{Z}^d$ with $|r|\le 2M$, we consider J a finite subset of $\mathbb{Z}\times\mathbb{Z}^d$ and for $\epsilon_J=(\epsilon_{j,k})_{(j,k)\in J}\in\{-1,1\}^J$ and T_{ϵ_J} the operator

$$T_{\epsilon_J}(\vec f)=\sum_{(j,k)\in J}\epsilon_{j,k}\langle\vec f|\vec\alpha^*_{\epsilon,j,k+r}\rangle\ \vec\gamma^*_{\eta,j,k}$$

Using again the $T(1)$ theorem, we see that $\|T_{\epsilon_J}\|_{CZO} \leq C$, so that $T_{\epsilon_J}(\vec{f}) \in X_0^d$ and

$$\int T_{\epsilon_J}(\vec{f}).\vec{g}\, dx = \sum_{(j,k)\in J} \epsilon_{j,k} \langle \vec{f}|\vec{\alpha}^*_{\epsilon,j,k+r}\rangle\ \langle \vec{g}|\vec{\gamma}^*_{\eta,j,k}\rangle \leq C\|\vec{f}\|_{X_0^d}\|\vec{g}\|_{Y^d}$$

Now, it is enough to choose $\epsilon_{j,k}$ as the sign of $\langle \vec{f}|\vec{\alpha}^*_{\epsilon,j,k+r}\rangle\ \langle \vec{g}|\vec{\gamma}^*_{\eta,j,k}\rangle$ and we may conclude.

Thus, Theorem 2 has been proved. $\diamond$

4. Examples.

We now give some examples of Calderón–Zygmund pairs of Banach spaces (according to Defiinition 2) :

a) **Lebesgue spaces :** $X = X_0 = L^p$ and $Y = Y_0 = L^q$ with $1 < p < +\infty$ and $1/p + 1/q = 1$.

b) **Lorentz spaces :** $X = X_0 = L^{p,r}$ and $Y = L^{q,\rho}$ with $1 < p < +\infty$, $1 \leq r < +\infty$, $1/p + 1/q = 1$ and $1/r + 1/\rho = 1$.

c) **Weighted Lebesgue spaces :** $X = X_0 = L^p(w\ dx)$ and $Y = Y_0 = L^q(w^{-\frac{1}{p-1}}\ dx)$ with $1 < p < +\infty$ and $1/p + 1/q = 1$, when the weight w belongs to the Muckenhoupt class $\mathcal{A}_p$.

d) **Morrey spaces :** We consider the Morrey space $\mathcal{L}^{\alpha,p}$ defined by

$$f \in \mathcal{L}^{\alpha,p} \Leftrightarrow \sup_{Q\in\mathcal{Q}} R_Q^\alpha \left(\frac{1}{|Q|}\int_Q |f(x)|^p\ dx\right)^{1/p} < \infty$$

We are interested in the set of parameters $1 < p < +\infty$ and $0 < \alpha \leq d/p$.

The Zorko space $\mathcal{L}_0^{\alpha,p}$ is the closure of $\mathcal{D}$ in $\mathcal{L}^{\alpha,p}$. Adams and Xiao [ADAX 11] have proved that $\mathcal{L}^{\alpha,p}$ is the bidual of $\mathcal{L}_0^{\alpha,p}$: $\mathcal{H}^{\alpha,q} = (\mathcal{L}_0^{\alpha,p})^*$ and $\mathcal{L}^{\alpha,p} = (\mathcal{H}^{\alpha,q})^*$ with $1/p + 1/q = 1$. One characterization of $\mathcal{H}^{\alpha,p}$ is the following one : $f \in \mathcal{H}^{\alpha,q}$ if and only if there is a sequence $(\lambda_n)_{n\in\mathbb{N}} \in l^1$ and a sequence of functions f_n and of cubes Q_n such that $f_n \in L^q$, f_n is supported in Q_n and $\|f_n\|_q \leq R_{Q_n}^{\alpha+d/q-d}$. The norm $\|f\|_{\mathcal{H}^{\alpha,q}}$ is then equivalent to $\inf_{(\lambda_n),(f_n),f=\sum\lambda_n f_n} \sum_{n\in\mathbb{N}} |\lambda_n|$.

Our Calderón–Zygmund pair is then $X = \mathcal{L}^{\alpha,p}$ and $Y = Y_0 = \mathcal{H}^{\alpha,q}$ with $1 < p < +\infty$, $0 < \alpha \leq d/p$ and $1/p + 1/q = 1$.

e) **Multipliers spaces :** We can build new examples from the former ones. Indeed, let X be a Banach space such that
i) we have the continuous embeddings : $X_1 \subset X \subset X_2$ for some Calderón–Zygmund pairs of Banach spaces (X_1, Y_1) and (X_2, Y_2)

iii) There is a Banach space A such that $\mathcal{D}$ is dense in A and the dual space A^* of A coincides with X with equivalence of norms
iii) Every Calderón–Zygmund operator may be extended as a bounded operator on X : $\|T(f)\|_X \leq C\|T\|_{CZO}\|f\|_X$.

Then, if X_0 is the closure of $\mathcal{D}$ in X and $Y = X_0^*$, (X, Y) is a Calderón–Zygmund pairs of Banach space (and $A = Y_0$).

This is easy to prove. First, let notice that every Calderón–Zygmund operator can be extended on X_2, hence de defined on X; the extra information is that it is bounded from X to X. Moreover, we have $\mathcal{D} \subset X_{1,0} \subset X_0$ with continuous embeddings, so that t every Calderón–Zygmund operator maps X_0 to X_0, hence by duality maps Y to Y. Moreover, from $X_{1,0} \subset X_0 \subset X_{2,0}$, we get $Y_2 \subset Y \subset Y_1$. We will conclude if we prove $A = Y_0$; but we see easily (since truncate and convolution operators are Calderón-Zygmund operators) that X_0 is *-weakly dense in X and that A is embedded into Y with equivalence of norms (due to hahn–Banach theorem). Thus, $A = Y_0$.

We may apply this to the space $X = X^{s,p}$ of pointwise multipliers from potential space $\dot{H}^s_p$ $(1 < p < +\infty,\ 0 < s < d/p)$:
i) we have the continuous embeddings for $p_1 > p$: $\mathcal{L}^{s,p_1} \subset X^{s,p} \subset \mathcal{L}^{s,p}$ (Fefferman-Phong inequality) [FEF 83]
iii) $X^{s,p}$ is the dual space of $Y^{s,q}$ defined by : $f \in Y^{s,q}$ if and only if there is a sequence $(\lambda_n)_{n\in\mathbb{N}} \in l^1$ and a sequence of functions f_n and g_n with $f_n \in \dot{H}^s_p$, $g_n \in L^q$, $\|f_n\|_{\dot{H}^s_p} \leq 1$ and $\|g_n\|_q \leq 1$. The norm $\|f\|_{Y^{s,q}}$ is then equivalent to $\inf_{(\lambda_n),(f_n),(g_n), f=\sum \lambda_n f_n g_n} \sum_{n\in\mathbb{N}} |\lambda_n|$.
iii) Every Calderón–Zygmund operator may be extended as a bounded operator on X : $\|T(f)\|_X \leq C\|T\|_{CZO}\|f\|_X$. This is due to a theorem of Verbitsky [MAZV 95].

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