A remark on the div-curl lemma.

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Abstract: We prove the div-curl lemma for a general class of functional spaces, stable under the action of Calderón–Zygmund operators. The proof is based on a variant of the renormalization of the product introduced by S. Dobyinsky, and on the use of divergence–free wavelet bases.

Introduction.

In 1992, Coifman, Lions, Meyer and Semmes published a paper [COILMS 92] where they gave a new interpretation of the compensated compactness introduced by Murat and Tartar [MUR 78]. They showed that the functions considered by Murat and Tartar had a greater regularity than expected: they belonged to the Hardy space $H^1$.

They gave a new version of the div-curl lemma of Murat and Tartar:

**Theorem 1**

If $1 < p < \infty$, $q = p/(p-1)$, $\vec{f} \in (L^p(\mathbb{R}^d))^d$ and $\vec{g} \in (L^q)^d$, then

$$\text{div } \vec{f} = 0 \text{ and curl } \vec{g} = \vec{0} \Rightarrow \vec{f}.\vec{g} \in H^1$$

There are many proofs of this result. We shall rely mainly on the proof by S. Dobyinsky, based on the renormalization of the product introduced in [DOB 92].

As pointed to me by Prof. Grzegorz Karch, it is easy to see that this result may be extended to a large class of functional spaces. For instance, we have the straightforward consequence of Theorem 1, for the case of weak Lebesgue spaces $L^{p,*}$ (better seen as Lorentz spaces $L^{p,\infty}$) and their preduals $L^{q,1}$:

**Corollary 1**:

If $1 < p < \infty$, $q = p/(p-1)$, $\vec{f} \in (L^{p,\infty}(\mathbb{R}^d))^d$ and $\vec{g} \in (L^{q,1})^d$, then

$$\text{div } \vec{f} = 0 \text{ and curl } \vec{g} = \vec{0} \Rightarrow \vec{f}.\vec{g} \in H^1$$

and

$$\text{div } \vec{g} = 0 \text{ and curl } \vec{f} = \vec{0} \Rightarrow \vec{f}.\vec{g} \in H^1$$
Proof: All we need is the projection operators that lead to the Helmholtz decomposition of a vector field: $\text{Id} = \mathbf{P} + \mathbf{Q}$ where $\mathbf{Q}$ is the projection onto irrotational vector fields:

$$\mathbf{Q} \tilde{\mathbf{v}} = \tilde{\nabla} \frac{1}{\Delta} \text{div} \, \tilde{\mathbf{v}}$$

and $\mathbf{P}$ the projection operator onto solenoidal vector fields. Those projection operators are matrix of singular integral operators and thus are bounded on Lebesque spaces $L^r$, $1 < r < \infty$, and, by interpolation, on Lorentz spaces spaces $L^{r,t}$, $1 < r < \infty$, $1 \leq t \leq +\infty$.

Let $\epsilon > 0$ such that $\epsilon < \min 1/p, 1/q$. We write $\frac{1}{p_+} = \frac{1}{p} + \epsilon$, $\frac{1}{p_-} = \frac{1}{p} - \epsilon$, $\frac{1}{q_+} = \frac{1}{q} + \epsilon$ and $\frac{1}{q_-} = \frac{1}{q} - \epsilon$. If $\vec{f} \in (L^{p,\infty}(\mathbb{R}^d))^d$, we can write, for every $A > 0$, $\vec{f} = \vec{\alpha}_A + \vec{\beta}_A$ with $\|\vec{\alpha}_A\|_{L^{p,\infty}} \leq CA\|\vec{f}\|_{L^{p,\infty}}$ and $\|\vec{\beta}_A\|_{L^{p,\infty}} \leq CA^{-1}\|\vec{f}\|_{L^{p,\infty}}$. If $\text{div} \, \vec{f} = 0$, we have moreover $\vec{f} = \mathbf{P}\vec{f} = \mathbf{P}\vec{\alpha}_A + \mathbf{P}\vec{\beta}_A$. On the other hand, if $\vec{g} \in (L^{q,1})^d$, we can write $\vec{g} = \sum_{j \in \mathbb{N}} \lambda_j \vec{g}_j$ with $\|\vec{g}_j\|_{L^{p-}} \leq 1$ and $\sum_{j \in \mathbb{N}} |\lambda_j| \leq C\|\vec{g}\|_{L^{q,1}}$. If $\text{curl} \, \vec{g} = 0$, we have moreover $\vec{g} = \mathbf{Q}\vec{g} = \sum_{j \in \mathbb{N}} \lambda_j \mathbf{Q}\vec{g}_j$. Let $A_j = \|\vec{g}_j\|_{L^{p-}}^{1/2} \|\vec{g}_j\|_{L^{q+}}^{-1/2}$. We then write

$$\vec{f} \cdot \vec{g} = \sum_{j \in \mathbb{N}} \lambda_j (\mathbf{P}\vec{\alpha}_A, \mathbf{Q}\vec{g}_j + \mathbf{P}\vec{\beta}_A, \mathbf{Q}\vec{g}_j)$$

and get (from the div-curl theorem of Coifman, Lions, Meyer and Semmes)

$$\|\vec{f} \cdot \vec{g}\|_{H^1} \leq C \sum_{j \in \mathbb{N}} |\lambda_j| (\|\mathbf{P}\vec{\alpha}_A\|_{L^{p-}} \|\mathbf{Q}\vec{g}_j\|_{L^{q+}} + \|\mathbf{P}\vec{\beta}_A\|_{L^{p+}} \|\mathbf{Q}\vec{g}_j\|_{L^{q-}})$$

$$\leq C' \|\vec{f}\|_{L^{p,\infty}} \sum_{j \in \mathbb{N}} |\lambda_j| (A_j \|\vec{g}_j\|_{L^{q+}} + A_j^{-1} \|\vec{g}_j\|_{L^{q-}})$$

$$= C' \|\vec{f}\|_{L^{p,\infty}} \sum_{j \in \mathbb{N}} |\lambda_j|$$

$$\leq C'' \|\vec{f}\|_{L^{p,\infty}} \|\vec{g}\|_{L^{q,1}}$$

The proof for the case $\text{div} \, \vec{g} = 0$ and $\text{curl} \, \vec{f} = 0$ is similar.

In this paper, we aim to find a general class of functional spaces for which the div-curl lemma still holds. As we may see from the proof of the Corollary 1, singular integral operators will play a key role in our result. In section 1, we shall introduce Calderón–Zygmund pairs of functional spaces which will allow us to prove such a general result. In section 2, we recall basics of divergence–free wavelet bases (as described in the book [LEM 02]). In section 3, we prove our main theorem. Then, in section 4, we give examples of Calderón–Zygmund pairs of functional spaces.


We begin by recalling the definition of a Calderón–Zygmund operator:
Definition 1 :

A singular integral operator is a continuous linear mapping from $D(\mathbb{R}^d)$ to $D'(\mathbb{R}^d)$ whose distribution kernel $K(x,y) \in D'(\mathbb{R}^d \times \mathbb{R}^d)$ (defined formally by the formula $Tf(x) = \int K(x,y)f(y) \, dy$) has its restriction outside the diagonal $x = y$ defined by a locally Lipschitz function with the following size estimates :

i) $\sup_{x \neq y} | K(x,y) | | x - y |^d < +\infty$

ii) $\sup_{x \neq y} | \nabla_x K(x,y) | | x - y |^{d+1} < +\infty$

iii) $\sup_{x \neq y} | \nabla_y K(x,y) | | x - y |^{d+1} < +\infty$

For such an operator $T$, we define

$$\|T\|_{SIO} = \|K(x,y)\|_{L^\infty(\Omega)} + \|\nabla_x K(x,y)\|_{L^\infty(\Omega)} + \|\nabla_y K(x,y)\|_{L^\infty(\Omega)}$$

where $K$ is the distribution kernel of $T$ and $\Omega = \mathbb{R}^d \times \mathbb{R}^d - \{(x,y) / x = y\}$

B) A Calderón–Zygmund operator is a singular integral operator $T$ which may be extended as a bounded operator on $L^2$ : $\sup_{\varphi \in D, \|\varphi\|_2 \leq 1} \|T(\varphi)\|_2 < +\infty$.

We define $CZO$ as the space of Calderón–Zygmund operators, endowed with the norm :

$$\|T\|_{CZO} = \|T\|_{L(L^2,L^2)} + \|T\|_{SIO}.$$

We may now define our main tool :

Definition 2 :

A Calderón–Zygmund pair of Banach spaces $(X,Y)$ is pair of Banach spaces such that :

i) we have the continuous embedding : $D(\mathbb{R}^d) \subset X \subset D'$ and $D(\mathbb{R}^d) \subset Y \subset D'$

iii) Let $X_0$ be the closure of $D$ in $X$; then the dual space $X_0^*$ of $X_0$ (i.e. the space of bounded linear forms on $X_0$) coincides with $Y$ with equivalence of norms :

A distribution $T$ belongs to $Y$ if and only if there exist a constant $C_T$ such that for all $\varphi \in D$ we have $\|T(\varphi)\|_{D',D} \leq C_T \|\varphi\|_X$

i)iiii) Let $Y_0$ be the closure of $D$ in $Y$; then the dual space $Y_0^*$ of $Y_0$ coincides with $X$ with equivalence of norms

ii) Every Calderón–Zygmund operator may be extended as a bounded operator on $X_0$ and on $Y_0$ : there exists a constant $C_0$ such that, for every $T \in CZO$ and every $\varphi \in D$, we have $T(\varphi) \in X_0 \cap Y_0$ and

$$\|T(\varphi)\|_X \leq C_0\|T\|_{CZO}\|\varphi\|_X \text{ and } \|T(\varphi)\|_Y \leq C_0\|T\|_{CZO}\|\varphi\|_Y$$

By duality, we find that every Calderón–Zygmund operator may be extended as a bounded operator on $X$ and $Y$ : if $T^*$ is defined by the formula

$$\langle T(\varphi) | \psi \rangle_{D',D} = \langle \varphi | T^*(\psi) \rangle_{D,D'},$$
then \( T \in CZO \) implies \( T^* \in CZO \) and we may define \( T(f) \) on \( X \) as the distribution \( \varphi \mapsto \langle f|T^*(\varphi)\rangle_{Y^*,Y_0} \). The two definitions of \( T \) coincides on \( X_0 \).

For \( m \in L^\infty \), the operator \( T_m : \varphi \mapsto m\varphi \) belongs to \( CZO \) (with kernel \( K(x,y) = m(x)\delta(x-y) \)). The stability of \( X \) and \( Y \) through multiplication by bounded smooth functions (with the inequalities \( \|mf\|_X \leq C_0 \|m\|_\infty \|f\|_X \) and \( \|mf\|_Y \leq C_0 \|m\|_\infty \|f\|_Y \)) shows that elements of \( X \) and \( Y \) are (complex) local measures and that \( X_0 \) and \( Y_0 \) are embedded into \( L^1_{loc} \).

Our main result is then the following one (to be proved in Section 3) :

**Theorem 2 :**

Let \((X,Y)\) be a Calderón–Zygmund pair of Banach spaces \((X,Y)\). If \( \vec{f} \in X^d_0 \) and \( \vec{g} \in Y^d \), then

\[
\text{div } \vec{f} = 0 \text{ and curl } \vec{g} = 0 \Rightarrow \vec{f} \cdot \vec{g} \in H^1
\]

and

\[
\text{div } \vec{g} = 0 \text{ and curl } \vec{f} = 0 \Rightarrow \vec{f} \cdot \vec{g} \in H^1
\]

**Remark :** The distribution \( \vec{f} \cdot \vec{g} \) is well-defined, since \( \vec{f} \in X^d_0 \) : if \( \varphi \in D \), then we have \( \varphi \vec{f} \in X^d_0 \) and \( \vec{g} \in (X^*_0)^d \).

### 2. Divergence–free wavelet bases.

In this section, we give a short review of properties of divergence-free wavelet bases. Wavelet theory was introduced in the 1980’s as an efficient tool for signal analysis. Orthonormal wavelet bases were first constructed by Y. Meyer [LEMM 86], G. Battle [BAT 87] and P.G. Lemarié-Rieusset; a major advance was done with the construction of compactly supported orthonormal wavelets by I. Daubechies [DAU 92]. Then bi-orthogonal bases were introduced by A. Cohen, I. Daubechies and J.C. Feauveau [COHDF 92]. Divergence-free wavelets were introduced by Battle and Federbush [BATF 95]. Compactly divergence-free wavelets were introduced by P.G. Lemarié–Rieusset [LEM 92]; they are not orthogonal wavelets [LEM 94], but have been explored for the numerical analysis of the Navier–Stokes equations [URB 95] [DER 06].

Let \( H_{\text{div}=0} \) and \( H_{\text{curl}=0} \) be defined as

\[
H_{\text{div}=0} = \{ \vec{f} \in (L^2)^d / \text{div } \vec{f} = 0 \} \quad \text{and} \quad H_{\text{curl}=0} = \{ \vec{f} \in (L^2)^d / \text{curl } \vec{f} = 0 \}.
\]

For a function \( \vec{f} \in (L^2)^d, j \in \mathbb{Z} \) and \( k \in \mathbb{Z}^d \), we define \( \vec{f}_{j,k} \) as \( \vec{f}_{j,k}(x) = 2^{jd/2}\vec{f}(2^jx - k) \). Let us recall the mains results of [LEM 92] (described as well in the book [LEM 02]). The idea is to begin with an Hilbertian basis of compactly supported wavelets, associated to
a multi-resolution analysis \((V_j)_{j \in \mathbb{Z}}\) of \(L^2(\mathbb{R})\). Associated to this multi-resolution analysis (with orthogonal projection operator \(\Pi_j\) onto \(V_j\)), there is a bi-orthogonal multi-resolution analysis \((V^+_j)\ (V^-_j)\) with projection \(\Pi^{\pm}_j\) onto \(V^-_j\) orthogonally to \(V^+_j\) such that \(\frac{d}{dx} \circ \Pi_j = \Pi_j \circ \frac{d}{dx}\).

Starting from this one-dimensional setting, we now consider a bi-orthogonal multi-resolution analysis of \((L^2(\mathbb{R}^d))^d\ (V_{j,1}, \ldots, V_{j,d})\) and \((V^+_j, \ldots, V^+_{d,j})\) where \(V_{j,k} = V_{j,k,1} \otimes \ldots \otimes V_{j,k,d}\) with \(V_{j,k,l} = V_j\) for \(k \neq l\) and \(V^*_j\). Let \(P_j\) be the projection operator onto \((V_{j,1}, \ldots, V_{j,d})\) orthogonally to \((V^*_j, \ldots, V^*_{j,d})\). Its adjoint \(P_j^*\) is the projection operator onto \((V^*_j, \ldots, V^*_{j,d})\) orthogonally to \((V_{j,1}, \ldots, V_{j,d})\). The point is that we have \(P_j(\nabla f) = \nabla (\Pi_j f)\) and \(\text{div} \ (P_j^* f) = \Pi_j^*(\text{div} \ f)\).

Those projection operators \(P_j\) and \(P_j^*\) can give an accurate description of \(H_{\text{div} = 0}\) and \(H_{\text{curl} = 0}\).

**Proposition 1 :** (Multi-resolution analysis for divergence-free or irrotational vector fields)

Let \(N \in \mathbb{N}\). Then there exists a compact set \(K_N \subset \mathbb{R}^d\) such that:

**A) Multi-resolution analysis :** There exists

*) functions \(\varphi_\xi, \varphi^*_\xi\) in \((L^2)^d\), \(1 \leq \xi \leq d\)

*) functions \(\psi_\chi, \psi^*_\chi\) in \((L^2)^d\), \(1 \leq \chi \leq 2d - 1\)

such that

i) the functions \(\varphi\xi, \varphi^*_\xi, \psi_\chi\) and \(\psi^*_\chi\) are supported in the compact \(K_N\)

ii) the functions \(\varphi_\xi, \varphi^*_\xi, \psi_\chi, \psi^*_\chi\) are of class \(C^N\)

iii) for \(l \in \mathbb{N}^d\) with \(\sum_{i=1}^d l_i \leq N\), we have \(\int x^l \psi_\chi \ dx = \int x^l \psi^*_\chi \ dx = 0\)

iv) for \(j, j', \in \mathbb{Z}, k, k' \in \mathbb{Z}^d\), \(\xi, \xi' \in \{1, \ldots, d\}, \) and \(\chi, \chi' \in \{1, \ldots, d(2d - 1)\}\)

\[
\int \varphi_\xi,j,k: \varphi^*_\xi',j,k' \ dx = \delta_{k,k'} \delta_{\xi,\xi'} \quad \text{and} \quad \int \psi_\chi,j,k: \psi^*_\chi',j',k' \ dx = \delta_{j,j'} \delta_{k,k'} \delta_{\chi,\chi'}
\]

v) The projection operators \(P_j\) can be defined on \((L^2)^d\) by

\[
P_j(\tilde{f}) = \sum_{k \in \mathbb{Z}^d} \sum_{1 \leq \xi \leq d} \langle \tilde{f}, \varphi^*_\xi,j,k \rangle \ \varphi_\xi,j,k.
\]

They are bounded on \((L^2)^d\) and satisfy

\[
P_j \circ P_{j+1} = P_{j+1} \circ P_j = P_j, \lim_{j \to -\infty} \|P_j \tilde{f}\|_2 = 0 \quad \text{and} \quad \lim_{j \to +\infty} \|\tilde{f} - P_j \tilde{f}\|_2 = 0.
\]

vi) The operators \(Q_j\) defined on \((L^2)^d\) by

\[
Q_j(\tilde{f}) = \sum_{k \in \mathbb{Z}^d} \sum_{1 \leq \chi \leq d(2d - 1)} \langle \tilde{f}, \psi^*_{\chi,j,k} \rangle \ \psi_{\chi,j,k}
\]
are bounded on \((L^2)^d\) and satisfy
\[
Q_j = P_{j+1} - P_j
\]
and
\[
\|\vec{f}\|_2 \approx \sqrt{\sum_{j \in \mathbb{Z}} \|Q_j \vec{f}\|_2^2} \approx \sqrt{\sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \sum_{1 \leq \chi \leq d(2^d-1)} |\langle \vec{f} | \vec{\psi}_{\chi,j,k}^* \rangle|^2}
\]

B) Irrotational vector fields: The projection operators \(P_j\) satisfy:
\[
\vec{f} \in (L^2)^d \text{ and } \text{curl } \vec{f} = 0 \Rightarrow \text{curl } P_j(\vec{f}) = 0
\]
Moreover, there exists

* \(2^d-1\) functions \(\vec{\gamma}_\eta \in (L^2)^d, 1 \leq \eta \leq 2^d-1\) with \(\text{curl } \vec{\gamma}_\eta = 0\)

* \(2^d-1\) functions \(\vec{\gamma}^*_\eta \in (L^2)^d, 1 \leq \eta \leq 2^d-1\)
such that

i) the functions \(\vec{\gamma}_\eta\) and \(\vec{\gamma}^*_\eta\) are supported in the compact \(K_N\)

ii) the functions \(\vec{\gamma}_\eta\) and \(\vec{\gamma}^*_\eta\) are of class \(C^N\)

iii) for \(l \in \mathbb{N}^d\) with \(\sum_{i=1}^d l_i \leq N\), we have \(\int x^l \vec{\gamma}_\eta \, dx = \int x^l \vec{\gamma}^*_\eta \, dx = 0\)

iv) for \(j, j', \in \mathbb{Z}, k, k' \in \mathbb{Z}^d\) and \(\eta, \eta' \in \{1, \ldots, 2^d-1\},\)
\[
\int \vec{\gamma}_{\eta,j,k} \cdot \vec{\gamma}^*_{\eta',j',k'} \, dx = \delta_{j,j'}\delta_{k,k'}\delta_{\eta,\eta'}
\]

vi) The operators \(S_j\) defined on \((L^2)^d\) by
\[
S_j(\vec{f}) = \sum_{k \in \mathbb{Z}^d} \sum_{1 \leq \eta \leq 2^d-1} \langle \vec{f} | \vec{\gamma}^*_\eta,j,k \rangle \vec{\gamma}_{\eta,j,k}
\]
are bounded on \((L^2)^d\) and satisfy
\[
\forall \vec{f} \in H_{\text{curl}=0} \quad S_j \vec{f} = Q_j \vec{f}
\]
and
\[
\forall \vec{f} \in H_{\text{curl}=0} \quad \|\vec{f}\| \approx \sqrt{\sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \sum_{1 \leq \eta \leq 2^d-1} |\langle \vec{f} | \vec{\gamma}^*_\eta,j,k \rangle|^2}
\]

C) Divergence-free vector fields: The projection operators \(P_j\) satisfy:
\[
\vec{f} \in (L^2)^d \text{ and } \text{div } \vec{f} = 0 \Rightarrow \text{div } P_j^*(\vec{f}) = 0
\]
Moreover, there exists

* \((d-1)(2^d-1)\) functions \(\vec{\alpha}_\epsilon \in (L^2)^d, 1 \leq \epsilon \leq (d-1)(2^d-1)\) with \(\text{div } \vec{\alpha}_\epsilon = 0\)

* \((d-1)(2^d-1)\) functions \(\vec{\alpha}^*_\epsilon \in (L^2)^d, 1 \leq \epsilon \leq (d-1)(2^d-1)\)
such that
i) the functions $\vec{\alpha}_\epsilon$ and $\vec{\alpha}_\epsilon^*$ are supported in the compact $K_N$
ii) the functions $\vec{\alpha}_\epsilon$ and $\vec{\alpha}_\epsilon^*$ are of class $C^N$
iii) for $l \in \mathbb{N}^d$ with $\sum_{i=1}^d l_i \leq N$, we have $\int x^l \vec{\alpha}_\epsilon \, dx = \int x^l \vec{\alpha}_\epsilon^* \, dx = 0$
iv) for $j, j', k, k'$ in $\mathbb{Z}^d$ and $\epsilon, \epsilon'$ in $\{1, \ldots, (d-1)(2^d-1)\}$,
\[ \int \vec{\alpha}_{\epsilon,j,k} \vec{\alpha}_{\epsilon',j',k'} \, dx = \delta_{j,j'} \delta_{k,k'} \delta_{\epsilon,\epsilon'} \]
v) The operators $R_j$ defined on $(L^2)^d$ by
\[ R_j(f) = \sum_{k \in \mathbb{Z}^d} \sum_{1 \leq \epsilon \leq (d-1)(2^d-1)} \langle f | \vec{\alpha}_{\epsilon,j,k}^* \rangle \vec{\alpha}_{\epsilon,j,k} \]
are bounded on $(L^2)^d$ and satisfy
\[ \forall f \in H_{\text{div}=0} \quad R_j f = Q_j f \]
and
\[ \forall f \in H_{\text{div}=0} \quad \|f\|_2 \approx \sqrt{\sum_j \sum_{k \in \mathbb{Z}^d} \sum_{1 \leq \epsilon \leq (d-1)(2^d-1)} |\langle f | \vec{\alpha}_{\epsilon,j,k}^* \rangle|^2} \]

We would like now to use those special functions with our spaces $X$ and $Y$. We begin with the following lemma :

**Lemma 1 :**

a) If $f \in X_0^d$, then $P_j f$ and $P_j^* f$ converge strongly to 0 in $X^d$ as $j \to -\infty$ and converge strongly to $f$ in $X^d$ as $j \to +\infty$.

b) If $f \in Y_0^d$, then $P_j f$ and $P_j^* f$ converge strongly to 0 in $Y^d$ as $j \to -\infty$ and converge strongly to $f$ in $Y^d$ as $j \to +\infty$.

c) If $f \in X^d$, then $P_j f$ and $P_j^* f$ converge $*$-weakly to 0 in $X^d$ as $j \to -\infty$ and converge $*$-weakly to $f$ in $X^d$ as $j \to +\infty$.

d) If $f \in Y^d$, then $P_j f$ and $P_j^* f$ converge $*$-weakly to 0 in $Y^d$ as $j \to -\infty$ and converge $*$-weakly to $f$ in $Y^d$ as $j \to +\infty$.

**Proof :** First, we check that the operators are well defined. If $f \in C^N$ has a compact support, then we may write $f = f\theta$, with $\theta \in \mathcal{D}$ equal to 1 on a neighborhood of the support of $f$. Thus, $f = T_f(\theta)$ and we find that $f \in X_0 \cap Y_0$. Thus, $\langle f | g \rangle_{X_0,Y}$ is well
Thus, we write \( \bar{f} \). Consider the operators on \( X^d \):

\[
P_j(f) = \sum_{k \in \mathbb{Z}^d} \sum_{1 \leq \xi \leq d} \langle f, \bar{\varphi}_{\xi,j,k}^\ast \rangle_{X,Y_0} \varphi_{\xi,j,k}
\]

and

\[
P_j^\ast(f) = \sum_{k \in \mathbb{Z}^d} \sum_{1 \leq \xi \leq d} \langle f, \bar{\varphi}_{\xi,j,k}^\ast \rangle_{X,Y_0} \varphi_{\xi,j,k}^\ast.
\]

We have \( \sup_{j \in \mathbb{Z}} \|P_j\|_{CZO} = \sup_{j \in \mathbb{Z}} \|P_j^\ast\|_{CZO} < +\infty \). Thus, those operators are equicontinuous on \( X^d \).

To prove a), we need to check the limits only on a dense subspace of \( X_0^d \). \( X_0 \) cannot be embedded into \( L^1 \) : if \( f \in \mathcal{D} \) with \( \hat{f}(0) \neq 0 \), then the Riesz transforms \( R_j f \) are not in \( L^1 \) but belong to \( X_0 \). It means that \( f \in \mathcal{D} \mapsto \|f\|_1 \) is not continuous for the \( X_0 \) norm. We may find a sequence of functions \( f_n \) such that \( \|f_n\|_X \) converge to 0 and \( \|f_n\|_1 = 1 \). Since \( |f_n| \) is Lipschitz and compactly supported, we can regularize \( f_n \) and find a sequence of smooth compactly supported functions \( f_{n,k} \) such that all the \( f_{n,k} \), \( k \in \mathbb{N} \), are supported in a compact neighborhood of the support of \( f_n \) and converge, as \( k \to +\infty \), uniformly to \( |f_n| \); then, we have convergence in \( X \) (since \( Y_0 \subset L^1_{\text{loc}} \)) and in \( L^1 \). Thus, we can find a sequence of functions \( f_n \) which are in \( \mathcal{D} \), with \( \int f_n \, dx = 1 \) and \( \lim_{n \to +\infty} \|f_n\|_X = 0 \). This gives that the set of function \( f \in \mathcal{D} \) with \( \int f \, dx = 0 \) is dense in \( X_0 \).

We now consider \( Q_j = P_{j+1} - P_j \) and \( Q_j^\ast = P_{j+1}^\ast - P_j^\ast \):

\[
Q_j(f) = \sum_{k \in \mathbb{Z}^d} \sum_{1 \leq \xi \leq d} \langle f, \bar{\psi}_{\xi,j,k} \rangle_{X,Y_0} \psi_{\xi,j,k}
\]

and

\[
Q_j^\ast(f) = \sum_{k \in \mathbb{Z}^d} \sum_{1 \leq \xi \leq d} \langle f, \bar{\psi}_{\xi,j,k} \rangle_{X,Y_0} \psi_{\xi,j,k}^\ast.
\]

If \( \bar{f} \in \mathcal{D}^d \) and \( \int \bar{f} \, dx = 0 \), we have \( \bar{f} = \sum_{1 \leq l \leq d} \partial_l \bar{f} \) for some \( \bar{f} \in \mathcal{D}^d \). Similarly, we have \( \psi_{\chi}^\ast = \sum_{1 \leq l \leq d} \partial_l \psi_{\chi,l}^\ast \) and \( \psi_{\chi} = \sum_{1 \leq l \leq d} \partial_l \psi_{\chi,l} \) for some compactly supported functions of class \( C^N \). Thus, we find that, for \( \bar{f} \in \mathcal{D}^d \) with \( \int \bar{f} \, dx = 0 \),

\[
\|P_{j+1} \bar{f} - P_j \bar{f}\|_X + \|P_{j+1}^\ast \bar{f} - P_j^\ast \bar{f}\|_X \leq C \min \left( \sum_{l=1}^d \|\partial_l \bar{f}\|_X 2^{j_l}, \sum_{i=1}^d \|\bar{f}_i\|_X 2^{-j} \right).
\]

Thus, \( P_j \bar{f} \) and \( P_j^\ast \bar{f} \) have strong limits in \( X_0^d \) when \( j \) goes to \( -\infty \) or \( +\infty \). If \( \bar{g} \in \mathcal{D}^d \), we write \( \bar{f} \in (L^2)^d \) and \( \bar{g} \in (L^2)^d \), and see that

\[
\lim_{j \to -\infty} \langle P_j \bar{f}, \bar{g} \rangle_{X,Y_0} = \lim_{j \to -\infty} \langle P_j^\ast \bar{f}, \bar{g} \rangle_{X,Y_0} = 0
\]
\[
\lim_{j \to +\infty} \langle P_j \vec{f} | \vec{g} \rangle_{X,Y_0} = \lim_{j \to +\infty} \langle P_j^* \vec{f} | \vec{g} \rangle_{X,Y_0} = \langle \vec{f} | \vec{g} \rangle_{X,Y_0}.
\]

Thus, we have, for \( \vec{f} \in \mathcal{D}^d \) with \( \int \vec{f} \, dx = 0 \),

\[
\lim_{j \to -\infty} \| P_j \vec{f} \|_X = \lim_{j \to -\infty} \| P_j^* \vec{f} \|_X = 0
\]

and

\[
\lim_{j \to +\infty} \| P_j \vec{f} - \vec{f} \|_X = \lim_{j \to +\infty} \| P_j^* \vec{f} - \vec{f} \|_X = 0
\]

Thus a) is proved. b) is proved in a similar way. By duality, we get c) and d). 

We may now consider the operators :

\[
R_j(\vec{f}) = \sum_{k \in \mathbb{Z}^d} \sum_{1 \leq \epsilon \leq (d-1)(2d-1)} \langle \vec{f} | \vec{\alpha}^*_{\epsilon,j,k} \rangle_{X,Y_0} \vec{\alpha}_{\epsilon,j,k}
\]

and

\[
S_j(\vec{f}) = \sum_{k \in \mathbb{Z}^d} \sum_{1 \leq \eta \leq 2d-1} \langle \vec{f} | \vec{\gamma}^*_{\eta,j,k} \rangle_{X,Y_0} \vec{\gamma}_{\eta,j,k}.
\]

From the identities \( \mathbb{P}^* R_j^* = \mathbb{P}^* Q_j \) and \( \mathbb{Q}^* S_j^* = \mathbb{Q}^* Q_j^* \) which are valid from \( \mathcal{D}^d \) to \( Y_0^d \), we find by duality that \( R_j \mathbb{P} = Q_j^* \mathbb{P} \) and \( S_j \mathbb{Q} = Q_j \mathbb{Q} \) on \( X^d \). To be able to use those identities, we shall need the following lemma :

**Lemma 2 :**

Let \( \vec{f} \in X^d \). Then :

i) \( \mathbb{P} \vec{f} = \vec{f} \iff \text{div} \vec{f} = 0 \)

ii) \( \mathbb{Q} \vec{f} = \vec{f} \iff \text{curl} \vec{f} = 0 \)

**Proof :** First, we check that \( f \in X \) and \( \Delta f = 0 \Rightarrow f = 0 \). Take \( \theta \in \mathcal{D} \) such that \( \theta \geq 0 \), and \( \theta \neq 0 \), and define \( \gamma = \frac{1}{(1+x^2)^{\frac{n+1}{2}}} * \theta \). Convolution with the kernel \( \frac{1}{(1+x^2)^{\frac{n+1}{2}}} \) is a Calderón–Zygmund operator, so we get that \( \gamma \in Y_0 \). Moreover, if \( g \) is a function such that \( (1+x^2)^{\frac{n+1}{2}} g \in L^\infty \), we find that \( g = \gamma^{-1} g \gamma = T_{\gamma^{-1} g}(\gamma) \), where the pointwise multiplication operator \( T_{\gamma^{-1} g} \) is a Calderón–Zygmund operator, so we get that \( g \in Y_0 \). This proves that \( X \subset S' \). Thus, if \( f \in X \) and \( \Delta f = 0 \), we find that \( f \) is a harmonic polynomial. Moreover \( \int f |f|^\gamma \, dx = \langle f | T_{\gamma^{-1} g}(\gamma) \rangle_{X,Y_0} \), hence the integral \( \int f |f|^\gamma \, dx \) must be finite, and \( f \) must be constant. As the smooth functions with vanishing integral are dense in \( Y_0 \), we find that the constant is equal to 0.

Now, we have for a distribution \( \vec{f} \) that

\[
\text{div} \vec{f} = 0 \iff \forall \vec{\varphi} \in \mathcal{D}^d \text{ with curl } \vec{\varphi} = 0, \quad \langle \vec{f} | \vec{\varphi} \rangle = 0;
\]
thus, we have on $X^d$ that $\text{div} \, \mathbf{f} = 0$. Similarly, we have for a distribution $\mathbf{f}$ that

$$\text{curl} \, \mathbf{f} = 0 \iff \forall \varphi \in \mathcal{D}^d \text{ with } \text{div} \, \varphi = 0, \quad \langle \mathbf{f} | \varphi \rangle = 0;$$

thus, we have on $X^d$ that $\text{curl} \, \mathbf{Q} \mathbf{f} = 0$.

Conversely, we start from the decomposition $I d = I P + I Q$ valid on $X^d$. If $\text{div} \, \mathbf{f} = 0$, then we find that $\mathbf{h} = \mathbf{f} - I P \mathbf{f} = I Q \mathbf{f}$ satisfies. $\text{div} \, \mathbf{h} = 0$ and $\text{curl} \, \mathbf{h} = 0$. But this implies that $\Delta \mathbf{h} = 0$, hence $\mathbf{h} = 0$. We prove similarly that $\text{curl} \, \mathbf{f} = 0$ implies that $\mathbf{f} = I Q \mathbf{f}$. ⋄

3. The proof of the div-curl lemma.

As in [LEM 02], we prove Theorem 2 by adapting the proof given by Dobyinsky [DOB 92]. This proof uses the renormalization of the product through wavelet bases.

If $\mathbf{f} \in X^d$, $\mathbf{g} \in Y^d$ and if moreover $\mathbf{f} \in X_0^d$ or $\mathbf{g} \in Y_0^d$, we use lemma 1 to get that, in the distribution sense, we have

$$\mathbf{f} \cdot \mathbf{g} = \lim_{j \to +\infty} P_j^* \mathbf{f} \cdot P_j \mathbf{g} - P_{-j}^* \mathbf{f} \cdot P_{-j} \mathbf{g}$$

and thus

$$\mathbf{f} \cdot \mathbf{g} = \sum_{j \in \mathbb{Z}} P_j^* \mathbf{f} \cdot Q_j \mathbf{g} + Q_j^* \mathbf{f} \cdot P_j \mathbf{g} + Q_j \mathbf{f} \cdot Q_j^* \mathbf{g}.$$ 

If moreover $\text{div} \, \mathbf{f} = 0$ and $\text{curl} \, \mathbf{g} = 0$, we use lemma 2 to get that

$$\mathbf{f} \cdot \mathbf{g} = \sum_{j \in \mathbb{Z}} P_j^* \mathbf{f} \cdot S_j \mathbf{g} + R_j \mathbf{f} \cdot P_j \mathbf{g} + R_j \mathbf{f} \cdot S_j \mathbf{g}.$$ 

We shall prove that the three terms

$$A(\mathbf{f}, \mathbf{g}) = \sum_{j \in \mathbb{Z}} P_j^* \mathbf{f} \cdot S_j \mathbf{g},$$

$$B(\mathbf{f}, \mathbf{g}) = \sum_{j \in \mathbb{Z}} R_j \mathbf{f} \cdot P_j \mathbf{g}$$

and

$$C(\mathbf{f}, \mathbf{g}) = \sum_{j \in \mathbb{Z}} R_j \mathbf{f} \cdot S_j \mathbf{g}$$

belong to $\mathcal{H}^1$.

We make the proof in the case $\mathbf{f} \in X_0^d$ (the proof is similar in the case $\mathbf{g} \in Y_0^d$). We first check that $A$ and $B$ map $(X_0)^d \times Y^d$ to $\mathcal{H}^1$: we use the duality of $H^1$ and $CMO$ (the
closure of $C_0$ in $BMO$ (see Coifman and Weiss [COIW 77] and Bourdaud [BOU 02]) and try to prove that the operators

$$A(f, h) = \sum_{j \in \mathbb{Z}} S_j^* (h P_j^* f)$$

and

$$B(f, h) = \sum_{j \in \mathbb{Z}} P_j^* (h R_j f)$$

map $(X_0)^d \times CMO$ to $(X_0)^d$.

In order to prove this, we shall prove that $A(., h)$ and $B(., h)$ are matrices of singular integral operators when $h \in D$ and that we have the estimates $\|A(., h)\|_{CZO} \leq C \|h\|_{BMO}$ and $\|B(., h)\|_{CZO} \leq C \|h\|_{BMO}$. For $B$, we may as well study the adjoint operator

$$B^*(f, h) = \sum_{j \in \mathbb{Z}} R_j^* (h P_j f)$$

First, we estimate the size of the kernels and of their gradients. The kernels $A_h(x, y)$ of $A(., h)$ and $B_h^*(x, y)$ of $B(., h)^*$ are given by

$$A_h(x, y) = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \sum_{1 \leq \xi \leq d} \sum_{l \in \mathbb{Z}^d} \sum_{1 \leq \eta \leq 2^d - 1} \tilde{\gamma}_{\eta, j, l}(x) \langle h \tilde{\varphi}_{\xi, j, k} | \tilde{\varphi}_{\eta, j, l} \rangle \bar{\varphi}_{\xi, j, k}(y)$$

and

$$B_h^*(x, y) = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \sum_{1 \leq \xi \leq d} \sum_{l \in \mathbb{Z}^d} \sum_{1 \leq \epsilon \leq (d-1)(2^d - 1)} \tilde{\alpha}_{\epsilon, j, l}(x) \langle h \tilde{\varphi}_{\xi, j, l} | \tilde{\alpha}_{\epsilon, j, k} \rangle \bar{\varphi}_{\xi, j, k}(y)$$

There are only a few terms that interact, because of the localization of the supports: if $K_N \subset B(0, M)$, then $\langle h \tilde{\varphi}_{\xi, j, k} | \tilde{\gamma}_{\eta, j, l} \rangle = \langle h \tilde{\varphi}_{\xi, j, l} | \tilde{\alpha}_{\epsilon, j, k} \rangle = 0$ if $|l - k| > 2M$. Let

$$C(h) = \sup_{j \in \mathbb{Z}, k \in \mathbb{Z}^d, 1 \leq \xi \leq d, \xi \in \mathbb{Z}^d, 1 \leq \eta \leq 2^d - 1} | \langle h \tilde{\varphi}_{\xi, j, k} | \tilde{\gamma}_{\eta, j, l} \rangle |$$

and

$$D(h) = \sup_{j \in \mathbb{Z}, k \in \mathbb{Z}^d, 1 \leq \xi \leq d, \xi \in \mathbb{Z}^d, 1 \leq \epsilon \leq (d-1)(2^d - 1)} | \langle h \tilde{\varphi}_{\xi, j, l} | \tilde{\alpha}_{\epsilon, j, k} \rangle |$$

Then we have

$$|A_h(x, y)| \leq \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} CC(h) 2^{jd} 1_{B(0, M)}(2^j x - k) 1_{B(0, 3M)}(2^j y - k)$$

and thus

$$|A_h(x, y)| \leq CC(h) \sum_{2^j |y-x| \leq 4M} 2^{jd} \leq C' C(h) |x - y|^{-d}$$
and similarly

$$|B_h(x, y)| \leq CD(h)|x - y|^{-d}.$$  

In the same way, we have

$$|\tilde{\nabla}_x A_h(x, y)| + |\tilde{\nabla}_y A_h(x, y)| \leq \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} CC(h)2^{j(d+1)}1_{B(0, M)}(2^j x - k)1_{B(0, 3M)}(2^j y - k)$$

and thus

$$|\tilde{\nabla}_x A_h(x, y)| + |\tilde{\nabla}_y A_h(x, y)| \leq CC(h)|x - y|^{-d-1}$$

and similarly

$$|\tilde{\nabla}_x B_h(x, y)| + |\tilde{\nabla}_y B_h(x, y)| \leq CD(h)|x - y|^{-d-1}.$$  

Moreover, the function \( \tilde{\varphi}_{\epsilon,j,k} \cdot \tilde{\gamma}_{\eta,j,l} \) is supported in \( B(2^{-j}k, M 2^{-j}) \), \( \| \tilde{\varphi}_{\epsilon,j,k} \cdot \tilde{\gamma}_{\eta,j,l} \|_\infty \leq C 2^j \)
and \( \int \tilde{\varphi}_{\epsilon,j,k} \cdot \tilde{\gamma}_{\eta,j,l} \, dx = 0 \) (since \( P_j \tilde{\varphi}_{\epsilon,j,k} = \tilde{\varphi}_{\epsilon,j,k} \) and \( Q_j \tilde{\gamma}_{\eta,j,l} = \tilde{\gamma}_{\eta,j,l} \)). Thus, we find that \( \| \tilde{\varphi}_{\epsilon,j,k} \cdot \tilde{\gamma}_{\eta,j,l} \|_{BMO} \leq C \), so that

$$C(h) \leq C\|h\|_{BMO}.$$  

We have similar estimates for \( \| \tilde{\varphi}_{\epsilon,j,l} \cdot \tilde{\alpha}_{\epsilon,j,k} \|_{BMO} \) (since \( P_j \tilde{\varphi}_{\epsilon,j,l} = \tilde{\varphi}_{\epsilon,j,l} \) and \( Q_j \tilde{\alpha}_{\epsilon,j,k} = \tilde{\alpha}_{\epsilon,j,k} \), and thus \( \int \tilde{\varphi}_{\epsilon,j,l} \cdot \tilde{\alpha}_{\epsilon,j,k} \, dx = 0 \)), and thus

$$D(h) \leq C\|h\|_{BMO}.$$  

Thus far, we have proven that \( A(\cdot, h) \) and \( B(\cdot, h) \) are singular integral operators. To prove \( L^2 \) boundedness, we use the \( T(1) \) theorem of David and Journé [DAVJ 84]. We've got to check that the operators are weakly bounded (in the sense of the WBP property), and to compute the images of the function \( f = 1 \) through the operators and through their adjoints.

Let \( x_0 \in \mathbb{R}^d \), \( r_0 > 0 \) and let \( \tilde{f} \) and \( \bar{g} \) be supported in \( B(x_0, r_0) \). We want to estimate

$$\langle A(\tilde{f}, \bar{g}) \rangle_{D', D} \text{ and } \langle B(\tilde{f}, \bar{g}) \rangle_{D', D}.$$

We have \( \langle A(\tilde{f}, \bar{g}) \rangle \leq \sum_{j \in \mathbb{Z}} A_j \) where

$$A_j = \sum_{k \in \mathbb{Z}^d} \sum_{1 \leq \xi \leq d} \sum_{l \in \mathbb{Z}^d} \sum_{1 \leq \eta \leq 2^d - 1} \left| \langle \bar{g} | \tilde{\gamma}_{\eta,j,l} \rangle \langle h \tilde{\varphi}_{\epsilon,j,k} \rangle \langle \tilde{f} | \tilde{\varphi}_{\epsilon,j,k} \rangle \right|$$

and similarly \( \langle B(\tilde{f}, \bar{g}) \rangle \leq \sum_{j \in \mathbb{Z}} B_j \) where

$$B_j = \sum_{k \in \mathbb{Z}^d} \sum_{1 \leq \xi \leq d} \sum_{l \in \mathbb{Z}^d} \sum_{1 \leq \epsilon \leq (d-1)(2^d - 1)} \left| \langle \bar{g} | \tilde{\varphi}_{\epsilon,j,l} \rangle \langle h \tilde{\varphi}_{\epsilon,j,k} \rangle \langle \tilde{f} | \tilde{\alpha}_{\epsilon,j,k} \rangle \right|$$

We have

$$A_j \leq C(h) \sum_{k \in \mathbb{Z}^d} \sum_{1 \leq \xi \leq d} \sum_{l \in \mathbb{Z}^d} \sum_{1 \leq \eta \leq 2^d - 1} \left| \langle \bar{g} | \tilde{\gamma}_{\eta,j,l} \rangle \langle \tilde{f} | \tilde{\varphi}_{\epsilon,j,k} \rangle \right|$$
which gives

\[
A_j \leq C(h) \sum_{k \in \mathbb{Z}^d} \sum_{1 \leq \xi \leq d} |\langle \tilde{f} | \tilde{\varphi}_{\xi,j,k} \rangle| \sum_{l \in \mathbb{Z}^d} \sum_{1 \leq \eta \leq 2^d - 1} |\langle \tilde{g} | \tilde{\gamma}_{\eta,j,l}^* \rangle| \leq CC(h) 2^{jd} \|\tilde{f}\|_1 \|\tilde{g}\|_1
\]

and

\[
A_j \leq C(h) \sqrt{\sum_{k \in \mathbb{Z}^d} \sum_{1 \leq \xi \leq d} |\langle \tilde{f} | \tilde{\varphi}_{\xi,j,k} \rangle|^2} \sqrt{\sum_{l \in \mathbb{Z}^d} \sum_{1 \leq \eta \leq 2^d - 1} |\langle \tilde{g} | \tilde{\gamma}_{\eta,j,l}^* \rangle|^2}
\]

and thus

\[
A_j \leq C'(h) \|S_j \tilde{g}\|_2 \|P_j^* \tilde{f}\|_2 \leq C''(h) 2^{-j} \|\nabla \tilde{g}\|_2 \|\tilde{f}\|_2.
\]

Finally, we get

\[
|\langle A(\tilde{f}, h) | \tilde{g}\rangle_{D^', D}| \leq CC(h)(\sum_{2^j r_0 \leq 1} 2^{jd}_0 \|\tilde{f}\|_2 \|\tilde{g}\|_2 + \sum_{2^j r_0 > 1} 2^{-j} \|\nabla \tilde{g}\|_2 \|\tilde{f}\|_2) \leq C'C(h)(\|\tilde{f}\|_2 + r_0 \|\nabla \tilde{f}\|_2)(\|\tilde{g}\|_2 + r_0 \|\nabla \tilde{g}\|_2).
\]

Similar computations (based on the inequality \(|R_j(\tilde{f})|_2 \leq C2^{-j} \|\nabla \tilde{f}\|_2\)) gives as well

\[
|\langle B(\tilde{f}, h) | \tilde{g}\rangle_{D^', D}| \leq CD(h)(\|\tilde{f}\|_2 + r_0 \|\nabla \tilde{f}\|_2)(\|\tilde{g}\|_2 + r_0 \|\nabla \tilde{g}\|_2).
\]

Thus, our operators satisfy the weak boundedness property.

We must now compute the distributions \(T(1)\) and \(T^*(1)\) when \(T\) is one component of the matrix of operators \(A(., h)\) or of \(B(., h)\). We must prove that, if \(\theta \in D\) is equal to 1 on a neighborhood of 0, if \(\tilde{\theta}_{t,R} = (\theta_{1,t,R}, \ldots, \theta_{d,t,R})\) with \(\theta_{k,t,R} = \delta_{k,t}\tilde{\theta}(\frac{x}{R})\) and if \(\tilde{\psi} \in D^d\) with \(\int \psi \ dx = 0\), then we have

\[
\lim_{R \to +\infty} \sum_{j \in \mathbb{Z}} S_j^*(hP_j^* \tilde{\theta}_{t,R}) \in (BMO)^d
\]

(the limit is taken in \((D'/\mathbb{R})^d\)) and similarly that

\[
\lim_{R \to +\infty} \sum_{j \in \mathbb{Z}} P_j(hS_j \tilde{\theta}_{t,R}) \in (BMO)^d
\]

\[
\lim_{R \to +\infty} \sum_{j \in \mathbb{Z}} P_j^*(hR_j \tilde{\theta}_{t,R}) \in (BMO)^d
\]

and

\[
\lim_{R \to +\infty} \sum_{j \in \mathbb{Z}} R_j^*(hP_j \tilde{\theta}_{t,R}) \in (BMO)^d
\]

To check that, we write \(\tilde{h}_t = (h_{1,t}, \ldots, h_{d,t})\) with \(h_{k,t} = \delta_{k,t}h\) and we consider \(\tilde{\psi} \in D^d\) with \(\int \psi \ dx = 0\). We have \(\sum_{j \in \mathbb{Z}} \|S_j(\tilde{\psi})\|_1 < +\infty\) and \(\|hP_j^* \tilde{\theta}_{t,R}\|_\infty \leq \|h\|_\infty \|\tilde{\theta}\|_\infty\) and thus we get by dominated convergence that
\[
\lim_{R \to +\infty} \int \tilde{\psi} \sum_{j \in \mathbb{Z}} S_j^* (hP_j^* \tilde{\theta}_{l,R}) \, dx = \sum_{j \in \mathbb{Z}} \int S_j \tilde{\psi} \tilde{h}_l \, dx.
\]

\(\sum_{j \in \mathbb{Z}} S_j \) is a matrix of Calderón–Zygmund operators \(T\) which satisfy \(T^*(1) = 0\), hence map \(\mathcal{H}^1\) to \(\mathcal{H}^1\), so that we find

\[
| \sum_{j \in \mathbb{Z}} \int S_j \tilde{\psi} \tilde{h}_l \, dx | \leq C\|h\|_{BMO} \|\tilde{\psi}\|_{\mathcal{H}^1},
\]

and thus \(\lim_{R \to +\infty} \sum_{j \in \mathbb{Z}} S_j^* (hP_j^* \tilde{\theta}_{l,R}) \in (BMO)^d\). Similar estimates prove that

\[
\lim_{R \to +\infty} \int \tilde{\psi} R_j^* (hP_j \tilde{\theta}_{l,R}) \, dx = \sum_{j \in \mathbb{Z}} \int R_j \tilde{\psi} \tilde{h}_l \, dx.
\]

and

\[
| \sum_{j \in \mathbb{Z}} \int R_j \tilde{\psi} \tilde{h}_l \, dx | \leq C\|h\|_{BMO} \|\tilde{\psi}\|_{\mathcal{H}^1},
\]

so that \(\lim_{R \to +\infty} \sum_{j \in \mathbb{Z}} R_j^* (hP_j \tilde{\theta}_{l,R}) \in (BMO)^d\).

On the other hand, we have

\[
| \int \tilde{\psi} P_j (hS_j \tilde{\theta}_{l,R}) \, dx | \leq C\|h\|_\infty \|P_j \tilde{\psi}\|_1 \|S_j \tilde{\theta}_{l,R}\|_\infty \leq C \|x\|_\infty \min(1, 2^j) \min(\|\theta\|_\infty, 2^{-j} R^{-1} \|\nabla \theta\|_\infty) = O(R^{-1/2})
\]

so that \(\lim_{R \to +\infty} \sum_{j \in \mathbb{Z}} P_j (hS_j \tilde{\theta}_{l,R}) = 0\). Similarly, we have \(\lim_{R \to +\infty} \sum_{j \in \mathbb{Z}} P_j^* (hR_j \tilde{\theta}_{l,R}) = 0\).

Thus, we have proved that \(A\) and \(B\) map \(X_0^d \times CMO\) to \(X_0^d\), and thus that \(A\) and \(B\) map \(X_0^d \times Y^d\) to \(\mathcal{H}^1\). We still have to deal with \(C(\vec{f}, \vec{g}) = \sum_{j \in \mathbb{Z}} R_j \vec{f} S_j \vec{g}\). We write

\[
C(\vec{f}, \vec{g}) = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \sum_{1 \leq \eta \leq 2^d-1} \sum_{l \in \mathbb{Z}^d} \sum_{1 \leq \epsilon \leq (d-1)(2^d-1)} \langle \vec{g} | \vec{\tilde{\gamma}}_{\eta,j,k}^* \rangle \langle \vec{f} | \vec{\alpha}_{\epsilon,j,l}^* \rangle \vec{\alpha}_{\epsilon,j,l} \vec{\gamma}_{\eta,j,k}
\]

We have \(\vec{\alpha}_{\epsilon,j,l} \vec{\gamma}_{\eta,j,k} = 0\) for \(|k-l| > 2M\) and \(\|\vec{\alpha}_{\epsilon,j,l} \vec{\gamma}_{\eta,j,k}\|_{\mathcal{H}^1} \leq C\) for \(|k-l| \leq 2M\). Thus, we are lead to prove that:

\[
\sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \sum_{1 \leq \eta \leq 2^d-1} \sum_{|l-k| \leq 2M} \sum_{1 \leq \epsilon \leq (d-1)(2^d-1)} \sum_{1 \leq \eta \leq 2^d-1} | \langle \vec{g} | \vec{\tilde{\gamma}}_{\eta,j,k}^* \rangle | \langle \vec{f} | \vec{\alpha}_{\epsilon,j,l}^* \rangle | \leq C\|\vec{f}\|_{X_0^d} \|\vec{g}\|_{Y^d}.
\]

For \(1 \leq \eta \leq 2^d-1, 1 \leq \epsilon \leq (d-1)(2^d-1)\) and \(r \in \mathbb{Z}^d\) with \(|r| \leq 2M\), we consider \(J\) a finite subset of \(\mathbb{Z} \times \mathbb{Z}^d\) and for \(\epsilon_J = (\epsilon_{j,k})_{(j,k) \in J} \in \{-1, 1\}^J\) and \(T_{\epsilon_J}\) the operator

\[
T_{\epsilon_J}(\vec{f}) = \sum_{(j,k) \in J} \epsilon_{j,k} \langle \vec{f} | \vec{\alpha}_{\epsilon_{j,k} r}^* \rangle \vec{\gamma}_{\eta_{j,k}}
\]

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Using again the $T(1)$ theorem, we see that $\|T_\varepsilon\|_{CZO} \leq C$, so that $T_\varepsilon(f) \in X_0^d$ and

$$\int T_\varepsilon(f) \tilde{g} \, dx = \sum_{(j,k) \in J} \varepsilon_{j,k} \langle \tilde{f}|\tilde{\alpha}_{\varepsilon,j,k+r}\rangle \langle \tilde{g}|\tilde{\gamma}_{\eta,j,k}\rangle \leq C\|\tilde{f}\|_{X_0^d}\|\tilde{g}\|_{Y^d}$$

Now, it is enough to choose $\varepsilon_{j,k}$ as the sign of $\langle \tilde{f}|\tilde{\alpha}_{\varepsilon,j,k+r}\rangle \langle \tilde{g}|\tilde{\gamma}_{\eta,j,k}\rangle$ and we may conclude.

Thus, Theorem 2 has been proved.

\[ \diamond \]

4. Examples.

We now give some examples of Calderón–Zygmund pairs of Banach spaces (according to Definition 2):

a) Lebesgue spaces: $X = X_0 = L^p$ and $Y = Y_0 = L^q$ with $1 < p < +\infty$ and $1/p + 1/q = 1$.

b) Lorentz spaces: $X = X_0 = L^{p,r}$ and $Y = Y_0 = L^{q,\rho}$ with $1 < p < +\infty$, $1 \leq r < +\infty$, $1/p + 1/q = 1$ and $1/r + 1/p = 1$.

c) Weighted Lebesgue spaces: $X = X_0 = L^p(w \, dx)$ and $Y = Y_0 = L^q(w^{-\frac{1}{p-1}} \, dx)$ with $1 < p < +\infty$ and $1/p + 1/q = 1$, when the weight $w$ belongs to the Muckenhoupt class $A_p$.

d) Morrey spaces: We consider the Morrey space $L^{\alpha,p}$ defined by

$$f \in L^{\alpha,p} \iff \sup_{Q \in \mathcal{Q}} R_Q^\alpha \left( \frac{1}{|Q|} \int_Q |f(x)|^p \, dx \right)^{1/p} < \infty$$

We are interested in the set of parameters $1 < p < +\infty$ and $0 < \alpha \leq d/p$.

The Zorko space $L_0^{\alpha,p}$ is the closure of $\mathcal{D}$ in $L^{\alpha,p}$. Adams and Xiao [ADAX 11] have proved that $L^{\alpha,p}$ is the bidual of $L_0^{\alpha,p}$: $H^{\alpha,q} = (L_0^{\alpha,p})^*$ and $L^{\alpha,p} = (H^{\alpha,q})^*$ with $1/p + 1/q = 1$. One characterization of $H^{\alpha,p}$ is the following one: $f \in H^{\alpha,q}$ if and only if there is a sequence $(\lambda_n)_{n \in \mathbb{N}}$ in $l^1$ and a sequence of functions $f_n$ and of cubes $Q_n$ such that $f_n \in L^q$, $f_n$ is supported in $Q_n$ and $\|f_n\|_q \leq R_{Q_n}^{\alpha+d/q-d}$. The norm $\|f\|_{H^{\alpha,q}}$ is then equivalent to $\inf_{(\lambda_n),(f_n))} \sum \lambda_n f_n \sum_{n \in \mathbb{N}} |\lambda_n|$. Our Calderón–Zygmund pair is then $X = L^{\alpha,p}$ and $Y = Y_0 = H^{\alpha,q}$ with $1 < p < +\infty$, $0 < \alpha \leq d/p$ and $1/p + 1/q = 1$.

e) Multipliers spaces: We can build new examples from the former ones. Indeed, let $X$ be a Banach space such that

i) we have the continuous embeddings: $X_1 \subset X \subset X_2$ for some Calderón–Zygmund pairs of Banach spaces $(X_1, Y_1)$ and $(X_2, Y_2)$
There is a Banach space $A$ such that $\mathcal{D}$ is dense in $A$ and the dual space $A^*$ of $A$ coincides with $X$ with equivalence of norms.

Every Calderón–Zygmund operator may be extended as a bounded operator on $X$:

$$\|T(f)\|_X \leq C\|T\|_{CZO}\|f\|_X.$$  

Then, if $X_0$ is the closure of $\mathcal{D}$ in $X$ and $Y = X_0^*$, $(X,Y)$ is a Calderón–Zygmund pairs of Banach space (and $A = Y_0$).

This is easy to prove. First, let notice that every Calderón–Zygmund operator can be extended on $X$, hence defined on $X$; the extra information is that it is bounded from $X$ to $X$. Moreover, we have $\mathcal{D} \subset X_{1,0} \subset X_0$ with continuous embeddings, so that every Calderón–Zygmund operator maps $X_0$ to $X_0$, hence by duality maps $Y$ to $Y$. Moreover, from $X_{1,0} \subset X_0 \subset X_{2,0}$, we get $Y_2 \subset Y \subset Y_1$. We will conclude if we prove $A = Y_0$; but we see easily (since truncate and convolution operators are Calderón-Zygmund operators) that $X_0$ is $\sigma$-weakly dense in $X$ and that $A$ is embedded into $Y$ with equivalence of norms (due to hahn–Banach theorem). Thus, $A = Y_0$.

We may apply this to the space $X = X^{s,p}$ of pointwise multipliers from potential space $\dot{H}^s_p$ ($1 < p < +\infty$, $0 < s < d/p$):

i) we have the continuous embeddings for $p_1 > p$ : $\mathcal{L}^{s,p_1} \subset X^{s,p} \subset \mathcal{L}^{s,p}$ (Fefferman-Phong inequality) [FEF 83]

iii) $X^{s,p}$ is the dual space of $Y^{s,q}$ defined by : $f \in Y^{s,q}$ if and only if there is a sequence $(\lambda_n)_{n \in \mathbb{N}} \in l^1$ and a sequence of functions $f_n$ and $g_n$ with $f_n \in \dot{H}^s_p$, $g_n \in L^q$, $\|f_n\|_{\dot{H}^s_p} \leq 1$ and $\|g_n\|_q \leq 1$. The norm $\|f\|_{Y^{s,q}}$ is then equivalent to $\inf_{(\lambda_n), (f_n), (g_n), f = \sum \lambda_n f_n g_n} \sum_{n \in \mathbb{N}} |\lambda_n|$.

iii) Every Calderón–Zygmund operator may be extended as a bounded operator on $X$:

$$\|T(f)\|_X \leq C\|T\|_{CZO}\|f\|_X.$$  This is due to a theorem of Verbitsky [MAZV 95].

References.


