Small data in an optimal Banach space for the parabolic-parabolic and parabolic-elliptic Keller-Segel equations in the whole space

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Abstract

We prove global existence and stability for the Keller–Segel equations with small initial values in the critical Morrey space.
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1 The Keller-Segel equations.

We consider two Keller-Segel equations defined on the whole space $\mathbb{R}^d$ (where $d \geq 1$). Those equations describe the evolution of the density u of a biological population submitted to the influence of a chemical agent with concentration φ [9].

The first model is called the parabolic-elliptic model :

$$\begin{cases} \partial_t u = \Delta u - \text{ div } (u\vec{\nabla}\varphi) \\ -\Delta\varphi = -\alpha\varphi + u \\ u = u_0 \text{ for } t = 0 \end{cases} \tag{PE}$$

The second model is called the parabolic-parabolic model and is given by

$$\begin{cases} \partial_t u = \Delta u - \text{ div } (u\vec{\nabla}\varphi) \\ \epsilon\partial_t\varphi = \Delta\varphi - \alpha\varphi + u \\ u = u_0 \text{ for } t = 0 \\ \varphi = 0 \text{ for } t = 0 \end{cases} \tag{PP_ϵ}$$

where $\epsilon > 0$.

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Note that in (PE) and (PP$_\epsilon$), the damping coefficient α is constant and non-negative :

$$\alpha \geq 0.$$

The value of α will have no great importance in our results.

We turn these equations into integro-differential equations on u, through the Duhamel formula : we consider the Green function G for the Laplacian, the heat kernel W_t and the Bessel kernel G_α defined as

$$G(x) = \begin{cases} \frac{1}{2\pi} \ln(\frac{1}{|x|}) & \text{if } d = 2 \\ \\ \frac{\Gamma(d/2)}{2(d-2)\pi^{d/2}} \frac{1}{|x|^{d-2}} & \text{if } d \geq 3 \end{cases} \tag{1}$$

$$W_t(x) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x|^2}{4t}} \tag{2}$$

$$G_\alpha(x) = \int_0^{+\infty} W_s(x) e^{-\alpha s} \, ds \tag{3}$$

so that, for $\psi \in \mathcal{S}(\mathbb{R}^d)$, we have

$$\Delta(G * \psi) = -\psi$$

$$\partial_t(W_t * \psi) = \Delta(W_t * \psi) \text{ and } \lim_{t \to 0} W_t * \psi = \psi$$

and

$$(-\Delta + \alpha \text{ Id})(G_\alpha * \psi) = \psi$$

If $\alpha = 0$, the parabolic-elliptic equation (PE) is then turned into the integral equation :

$$u = W_t * u_0 - \int_0^t \text{div } W_{t-s} * (u(u * \vec{\nabla} G)) \, ds \tag{IPE}$$

and the parabolic-parabolic equation (PP$_\epsilon$) is turned into the equation

$$u = W_t * u_0 - \int_0^t \text{div } W_{t-s} * (u(\frac{1}{\epsilon} \int_0^s \vec{\nabla} W_{\frac{s-\sigma}{\epsilon}} * u \, d\sigma)) \, ds \tag{IPP_ϵ}$$

If $\alpha > 0$, the parabolic-elliptic equation (PE) is then turned into the integral equation :

$$u = W_t * u_0 - \int_0^t \text{div } W_{t-s} * (u(u * \vec{\nabla} G_\alpha)) \, ds \tag{IPE_α}$$

and the parabolic-parabolic equation (PP_ϵ) is turned into the equation

$$u = W_t * u_0 - \int_0^t \operatorname{div} W_{t-s} * (u(\frac{1}{\epsilon}\int_0^s e^{-\alpha\frac{s-\sigma}{\epsilon}} \vec{\nabla} W_{\frac{s-\sigma}{\epsilon}} * u \, d\sigma)) \, ds \quad (IPP_{\alpha,\epsilon})$$

We are going to involve the four equations in a simultaneous study. As a matter of fact, we may write each equation in the form

$$u = W_t * u_0 - \int_0^t \operatorname{div} W_{t-s} * (u \, L_{\alpha,\epsilon}(u)) \, ds \tag{4}$$

where the linear operator $L_{\alpha,\epsilon}$ will satisfy size estimates independent of the specific parameters ϵ and α of the equation.

Proposition 1 (Size estimates for $L_{\alpha,\epsilon}$)
Let $M_* u$ be the time-variable Hardy–Littlewood maximal function :

$$M_* u(t,x) = \sup_{r>0} \frac{1}{2r} \int_{t-r}^{t+r} |u(s,x)| \, ds \tag{5}$$

and, for $0 < r < d$, let $I_r u$ be the space-variable Riesz potential of u

$$I_r u(t,x) = \frac{\Gamma((d-r)/2)}{\pi^{d/2} 2^r \Gamma(r/2)} \int \frac{1}{|x-y|^{d-r}} u(t,y) \, dy \tag{6}$$

Let $L_{\alpha,\epsilon}$ be the operator

$$L_{\alpha,\epsilon} u(t,x) = \frac{1}{\epsilon} \int_0^t e^{-\alpha\frac{t-\sigma}{\epsilon}} \vec{\nabla} W_{\frac{t-\sigma}{\epsilon}} * u \, d\sigma \text{ if } \alpha \geq 0 \text{ and } \epsilon > 0,$$

$$L_{\alpha,\epsilon} u(t,x) = u * \vec{\nabla} J_\alpha \text{ if } \epsilon = 0 \text{ and } \alpha > 0,$$

$$L_{\alpha,\epsilon} u(t,x) = u * \vec{\nabla} G \text{ if } \epsilon = \alpha = 0.$$

Then there exists a constant C_0 which does not depend on α nor on ϵ such that :

$$|L_{\alpha,\epsilon} u(t,x)| \leq C_0 \; I_1(M_* u)(t,x) \tag{7}$$

The proof of the proposition relies on two classical lemmas :

Lemma 1
If ω is a radially decreasing function on $\mathbb{R}^n$ and f a locally integrable function, then

$$|\int_{\mathbb{R}^n} \omega(x-y) f(y) \, dy| \leq \|\omega\|_1 \sup_{r>0} \frac{1}{|B(0,r)|} \int_{|y|<r} |f(x-y)| \, dy \tag{8}$$

or equivalently

$$|\omega * f| \leq \|\omega\|_1 M_f \tag{9}$$

where M_f is the Hardy–Littlewood maximal function of f.

Lemma 2
The function $e^{-|x|^2}$ *belongs to the Schwartz class. Therefore, we have for every* $\beta \in \mathbb{N}^d$

$$\sup_{x\in\mathbb{R}^d, t>0} (\sqrt{t} + |x|)^{d+|\beta|} |\partial_x^\beta W_t(x)| < \infty. \tag{10}$$

Lemma 1 is classical (see Grafakos[6] for instance). Lemma 2 is obvious. Using Lemma 2, we obtain, for $\epsilon > 0$,

$$|L_{\alpha,\epsilon} u(t,x)| \leq C_1 \frac{1}{\epsilon} \int_0^t \int \frac{1}{(\sqrt{\frac{t-\sigma}{\epsilon}} + |x-y|)^{d+1}} |u(\sigma,y)| \; dy \; d\sigma$$

where $C_1 = \sup_{x\in\mathbb{R}^d} (1+|x|)^{d+1} |\vec{\nabla} W_1(x)|$. Using Fubini's theorem, we first integrate with respect to $d\sigma$ and use Lemma 1 to get :

$$\begin{aligned} |L_{\alpha,\epsilon} u(t,x)| &\leq C_1 \int \Big(\int_{\mathbb{R}} \frac{d\sigma}{\epsilon((\sqrt{\frac{\sigma}{\epsilon}} + |x-y|)^{d+1}} \Big) M_*(t,y) \; dy \\ &= C_1 C_2 \int \frac{1}{|x-y|^{d-1}} M_* u(t,y) \; dy \end{aligned}$$

where $C_2 = \int_{\mathbb{R}} \frac{d\sigma}{(1+\sqrt{\sigma})^{d+1}}$. This proves inequality (7) for $\epsilon > 0$.

When $\epsilon = 0$, we write $|u(t,x)| \leq M_* u(t,x)$ and thus

$$|L_{0,0} u(t,x)| \leq \int |\vec{\nabla} G(x-y)| M_* u(t,y) \; dy \leq C_0 \; I_1(M_* u)(t,x).$$

For $\alpha > 0$, we write

$$\begin{aligned} |L_{\alpha,0} u(t,x)| &\leq \int \Big(\int_0^{+\infty} |\vec{\nabla} W_s(x-y)| ds \Big) M_* u(t,y) \; dy \\ &\leq C_1 \int \Big(\int_0^{+\infty} \frac{ds}{(\sqrt{s} + |x-y|)^{d+1}} \Big) M_* u(t,y) \; dy \\ &= C_1 C_2 \int \frac{1}{|x-y|^{d-1}} M_* u(t,y) \; dy. \end{aligned}$$

Thus Proposition 1 is proved.

2 Main results

We may now state our main result. We define the Morrey space $\dot{M}^1_{d/2}(\mathbb{R}^d)$ as the space of locally finite measures $d\mu$ such that

$$\sup_{x\in\mathbb{R}^d, r>0} r^{2-d} \int_{B(x,r)} d|\mu(y)| < +\infty. \tag{11}$$

This space is endowed with the norm

$$\|d\mu\|_{\dot M^1_{d/2}(\mathbb{R}^d)} = \sup_{x\in\mathbb{R}^d, r>0} r^{2-d} \int_{B(x,r)} d|\mu(y)|.$$

We shall see in Section 4 why this space is optimal for the search of global solutions to the Keller-Segel equations.

We shall use another Morrey-space, based on the weak Lebesgue spaces $L^{p,*}$ (or Marcinkiewicz spaces). In secion 3, we shall recall basic facts on Marcinkiewicz spaces. For $1 < p < +\infty$, the space $L^{p,*}$ is a Banach space. For $1/2 < \beta < 1$, we shall use the Morrey–Marcinkiewics space $\dot M^{2/(2-\beta),*}_{d/(2-\beta)}(\mathbb{R}^d)$. This space is defined as the space of measurable functions f that are locally in $L^{2/(2-\beta),*}$ and that are such that

$$\sup_{x\in\mathbb{R}^d, r>0} r^{(2-d)(1-\frac{\beta}{2})} \|\chi_{B(x,r)} f\|_{L^{d/(2-\beta),*}} < +\infty$$

where $\chi_{B(x,r)}$ is the characteristic function of the ball $B(x,r)$. This space is endowed with the norm

$$\|f\|_{\dot M^{2/(2-\beta),*}_{d/(2-\beta)}(\mathbb{R}^d)} = \sup_{x\in\mathbb{R}^d, r>0} r^{(2-d)(1-\frac{\beta}{2})} \|\chi_{B(x,r)} f\|_{L^{d/(2-\beta),*}}.$$

We associate to this space the space

$$E_\beta = \{u(t,x) \ / \ \sup_{t>0} t^{\beta/2} u(t,x) \in \dot M^{2/(2-\beta),*}_{d/(2-\beta)}\}.$$

The space E_β is normed with

$$\|u\|_{E_\beta} = \|\sup_{t>0} t^{\beta/2} u(t,x)\|_{\dot M^{2/(2-\beta),*}_{d/(2-\beta)}}.$$

We shall see that, for $1/2 < \beta < 2$ and $u_0 \in \dot M^1_{d/2}(\mathbb{R}^d)$, we have

$$W_t * u_0 \in E_\beta.$$

Our result is then the following :

Theorem 1 (Keller-Segel equations)
A) The operators $(f,g) \mapsto B_{\alpha,\epsilon}(f,g)$ where

$$B_{\alpha,\epsilon}(f,g)(t,x) = \int_0^t W_{t-s} * (f\, L_{\alpha,\epsilon} g)\ ds$$

are equicontinuous on E_β when $1/2 < \beta < 1$:

$$\|B_{\alpha,\epsilon}(f,g)\|_{E_\beta} \le C_0 \|f\|_{E_\beta} \|g\|_{E_\beta}$$

for a constant C_0 which does not depend on α nor on ϵ.
B) There exists a positive $\delta_0 > 0$ such that, for every $u_0 \in \dot{M}^1_{d/2}(\mathbb{R}^d)$ with $\|u_0\|_{\dot{M}^1_{d/2}((\mathbb{R}^d)} < \delta_0$, for every $\alpha \geq 0$ and every $\epsilon \geq 0$, the Picard iterates

$$v_{\alpha,\epsilon,0} = W_t * u_0 \text{ and } v_{\alpha,\epsilon,n+1} = v_0 - \int_0^t \, div\, W_{t-s} * (v_{\alpha,\epsilon,n} L_{\alpha,\epsilon}(v_{\alpha,\epsilon,n}))\; ds$$

converge in the E_β norm to a solution of the Keller-Segel equation $u_{\alpha,\epsilon} = W_t * u_0 - \int_0^t \, div\, W_{t-s} * (u_{\alpha,\epsilon}\; L_{\alpha,\epsilon} u_{\alpha,\epsilon})\; ds$.
C) Moreover, when ϵ goes to 0, the solution $u_{\alpha,\epsilon}$of the parabolic-parabolic problem converges in the E_β norm to the solution $u_{\alpha,0}$ of the parabolic-elliptic problem.

This theorem is a generalization of the existence theorems for small data proved by various authors in the setting of scaling invariant spaces : Lebesgue spaces [5], weak Lebesgue spaces [10, 12], Sobolev spaces [11], Besov spaces [8], pseudo-measures [3]. It is as well a generalization of the stability theorems proved by Biler and Brandolese [2] and Raczynski [14] (see the discussion in Section 4).

3 Morrey spaces

Let us recall the definition of Morrey spaces:

Definition 1 (Morrey spaces)
If $1 < p \leq q < +\infty$, the Morrey space $\dot{M}^p_q(\mathbb{R}^d)$ is the space of measurable functions f that are locally in L^p and that are such that

$$\sup_{x\in\mathbb{R}^d, r>0} r^{d(\frac{p}{q}-1)} \int_{B(x,r)} |f(y)|^p \; dy < +\infty.$$

This space is endowed with the norm

$$\|f\|_{\dot{M}^p_q(\mathbb{R}^d)} = \sup_{x\in\mathbb{R}^d, r>0} \big(r^{d(\frac{p}{q}-1)} \int_{B(x,r)} |f(y)|^p \; dy\big)^{1/p}.$$

If $1 \leq q < +\infty$, the Morrey space $\dot{M}^1_q(\mathbb{R}^d)$ is the space of locally finite measures $d\mu$ such that

$$\sup_{x\in\mathbb{R}^d, r>0} r^{d(\frac{1}{q}-1)} \int_{B(x,r)} d|\mu(y)| < +\infty.$$

This space is endowed with the norm

$$\|d\mu\|_{\dot{M}^1_q(\mathbb{R}^d)} = \sup_{x\in\mathbb{R}^d, r>0} r^{d(\frac{1}{q}-1)} \int_{B(x,r)} d|\mu(y)|$$

We have, for $p_0 \leq p_1 \leq q$, $1 < q$,, $L^q = \dot{M}_q^q \subset \dot{M}_q^{p_1} \subset \dot{M}_q^{p_0}$.

Instead of the L^p norm, it is sometimes useful to work in the weak Lebesgue space, or Marcinkiewicz space, $L^{p,*}$. Let us recall some basic facts abour Marcinkiewicz spaces. The space $L^{1,*}$ is the space of measurable functions f such that

$$\mathcal{N}(f) = \sup_{\lambda > 0} \lambda \left| \{ x \in \mathbb{R}^d \ / \ |f(x)| > \lambda \} \right| < +\infty. \tag{12}$$

We have obviously $L^1 \subset L^{1,*}$ and $\mathcal{N}(f) \leq \|f\|_1$. However, $L^{1,*}$ is not a space of distributions and its elements may be functions that are not locally integrable (such as $f(x) = \frac{1}{|x|^d}$). Moreover, $\mathcal{N}$ is not a norm (it is not convex).

For $1 < p < +\infty$, the Marcinkiewicz space $L^{p,*}$ is the space of measurable functions f such that $|f|^p \in L^{1,*}$. in that case, the functions are locally integrable : we have $L^{p,*} = [L^1, L^\infty]_{[1-\frac{1}{p},\infty]}$ and for two positive constants A_p and B_p, we have $A_p \|f\|_{[L^1,L^\infty]_{[1-\frac{1}{p},\infty]}} \leq \left(\mathcal{N}(|f|^p)\right)^{1/p} \leq B_p \|f\|_{[L^1,L^\infty]_{[1-\frac{1}{p},\infty]}}$. Thus, $L^{p,*}$ is a normed space, when we define the norm as $\|f\|_{L^{p,*}} = \|f\|_{[L^1,L^\infty]_{[1-\frac{1}{p},\infty]}}$.

Definition 2 (Morrey–Marcinkiewicz spaces)
If $1 < p \leq q < +\infty$, the Morrey–Marcinkiewicz space $\dot{M}_q^{p,*}(\mathbb{R}^d)$ is the space of measurable functions f that are locally in $L^{p,*}$ and that are such that

$$\sup_{x \in \mathbb{R}^d, r > 0} r^{d(\frac{1}{q} - \frac{1}{p})} \|\chi_{B(x,r)} f\|_{L^{p,*}} < +\infty$$

where $\chi_{B(x,r)}$ is the characteristic function of the ball $B(x,r)$. This space is endowed with the norm

$$\|f\|_{\dot{M}_q^{p,*}(\mathbb{R}^d)} = \sup_{x \in \mathbb{R}^d, r > 0} r^{d(\frac{1}{q} - \frac{1}{p})} \|\chi_{B(x,r)} f\|_{L^{p,*}}.$$

If $1 \leq q < +\infty$, the Morrey–Marcinkiewicz space $\dot{M}_q^{1,*}(\mathbb{R}^d)$ is the space of measurable functions f that are locally in $L^{1,*}$ and that are such that

$$\mathcal{N}_{*,q}(f) = \sup_{x \in \mathbb{R}^d, r > 0} r^{d(\frac{1}{q} - 1)} \mathcal{N}(\chi_{B(x,r)} f) < +\infty$$

Note that $\dot{M}_q^{q,*} = L^{q,*}$.

4 Optimal space for solving Keller-Segel equations

When dealing with the Keller-Segel equations on the whole space, one always try and use the symmetries of the equations :

- since the coefficients of the equations are constant, the equations are translation invariant : we find that if (u, φ) is a solution to (PP) or(PP_ϵ) with initial value u_0, then $(u(t, x - x_0), \varphi(t, x - x_0))$ is a solution of the same equation with initial value $u_0(x - x_0)$
- when $\alpha = 0$, the equations are scale invariant : if (u, φ) is a solution to (PP) or(PP_ϵ) with initial value u_0, then $(\lambda^2 u(\lambda^2 t, , \lambda x), \varphi(\lambda^2 t, , \lambda x))$ is a solution of the same equation with initial value $\lambda^2 u_0(\lambda x)$.

This explains the important litterature on Keller-Segel equations with data in scale invariant spaces such as $L^{d/2}$ [5], $L^{d/2,*}$ [10, 12], $\dot{H}^{d/p-2,p}$ [11], $\dot{B}^{-2+d/p}_{p,\infty}$ [8], $\mathcal{PM}^1$ [3]. However, there is another important feature of the Keller-Segel equation that should be underlined :

- if u_0 is non-negative, then u remains non-negative.

This property is important, as the equations aim to describe the density of a biological population. Focusing on this property, we may identify a good candidate for optimality in the search of global solutions :

Proposition 2 (Homogeneous shift–invariant Banach spaces)
A) Let E be a Banach space of tempered distributions on $\mathbb{R}^d$: $E \subset \mathcal{S}'(\mathbb{R}^d)$ (continuous embedding). If the norm of E is shift-invariant ($\|f(x - x_0)\|_E = \|f\|_E$) and homogeneous ($\|f(\lambda x)\|_E = \lambda^\gamma \|f\|_E$ for every $\lambda > 0$) with homogeneity exponent $\gamma < 0$, then E is continuously embedded into the homogeneous Besov space $\dot{B}^\gamma_{\infty,\infty}$.
B) If $u_0 \in \dot{B}^\gamma_{\infty,\infty}$ is non-negative and $-d \leq \gamma < 0$, then $u_0 \in \dot{M}^1_{d/|\gamma|}$ and there exists two positive constants A_γ and B_γ such that

$$A_\gamma \|u_0\|_{\dot{M}^1_{d/|\gamma|}} \leq \|u_0\|_{\dot{B}^\gamma_{\infty,\infty}} \leq B_\gamma \|u_0\|_{\dot{M}^1_{d/|\gamma|}} \tag{13}$$

The proposition is easily proved by using the characterization of the Besov space $\dot{B}^\gamma_{\infty,\infty}$ (with negative regularity exponent γ) through the heat kernel (see [13] for instance) :

- f belongs to $\dot{B}^\gamma_{\infty,\infty}$ if and only if $\sup_{t>0} t^{|\gamma|/2} \|W_t * f\|_\infty < +\infty$

- the norm of f in $\dot B^\gamma_{\infty,\infty}$ is equivalent to $\sup_{t>0} t^{|\gamma|/2}\|W_t * f\|_\infty < +\infty$.

We know that $|\langle f|W_1\rangle| \le C\|f\|_E$ for all $f\in E$ (since $E\subset \mathcal{S}'(\mathbb{R}^d)$). We have $W_t * f = \langle f(\sqrt{t}y+x)|W_1(-y)\rangle$, hence $\|W_t * f\|_\infty \le C\|f\|_E t^{\gamma/2}$. Thus A) is proved.

B) is easily checked as well. The existence of B_γ is a consequence of $\dot M^1_{d/|\gamma|}\subset \dot B^\gamma_{\infty,\infty}$ (since $\dot M^1_{d/|\gamma|}$ is a homogeneous shift–invariant Banach space). The existence of A_γ is easy as well : if u_0 is non negative, we have $u_0 = d\mu$ for a locally finite non-negative measure μ. We then write

$$\int_{B(x,r)} d\mu \le \frac{r^d\ u_0 * W_{r^2}(x)}{\inf_{|y|<1} W_1(y)} \le C r^{d+\gamma}\|u_0\|_{\dot B^\gamma_{\infty,\infty}}.$$

Thus, Proposition 2 is proved.

As a conclusion, we see that, in order to solve the Keller-Segel equations, one is lead to work with an initial data u_0 which belongs to a shift-invarariant Banach space whose norm is homogeneous with exponent -2, thus with $u_0\in \dot B^{-2}_{\infty,\infty}$; since we work with non-negative data (densities), we find that u_0 belongs to $\dot M^1_{d/2}$ and that smallness in the norm of u_0 in any other homogeneous shift-invariant Banach space will imply the smallness of u_0 in $\dot M^1_{d/2}$ as well.

5 Maximal functions and Morrey spaces

For a locally integrable function f, the Hardy-Littlewood maximal function of f is defined as $M_f(x) = \sup_{r>0} \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|dy$. We have the same definition for a locally finite measure $d\mu$: $M_{d\mu}(x) = \sup_{r>0}\frac{1}{|B(x,r)|}\int_{B(x,r)} d|\mu(y)|$.

We start from the well-known boundedness of the maximal function on L^p : if $1<p\le\infty$, $\|M_f\|_p \le C_p\|f\|_p$, while $\mathcal{N}(M_f)\le C_1\|f\|_1$ (see [6]). By real interpolation, one find that, for $1<p<+\infty$, $\|M_f\|_{L^{p,*}} \le C'_p\|f\|_{L^{p,*}}$. From those inequalities, one gets similar inequalities for Morrey or Morrey–Marcinkiewicz spaces :

Proposition 3 (Maximal functions and Morrey spaces)
A) For $1<p\le q<+\infty$, we have

$$\|M_f\|_{\dot M^p_q} \le C_{p,q}\|f\|_{\dot M^p_q} \text{ and } \|M_f\|_{\dot M^{p,*}_q} \le C_{p,q}\|f\|_{\dot M^{p,*}_q} \tag{14}$$

B) For $1\le q<+\infty$, we have

$$\mathcal{N}_{*,q}(M_{d\mu}) \le C_q\|d\mu\|_{\dot M^1_q} \tag{15}$$

The proof is quite direct. We just have to estimate $\|\chi_{B(x,r)} M_f\|_p$ or $\|\chi_{B(x,r)} M_f\|_{L^{p,*}}$ if $p > 1$ or $\mathcal{N}(\chi_{B(x,r)} M_f)$ if $p = 1$. We write $f_1 = f\chi_{B(x,3r)}$ and $f_2 = f - f_1$. We have $M_f \leq M_{f_1} + M_{f_2}$. Moreover, we have

$$\chi_{B(x,r)} M_{f_2}(y) \leq C r^{-d/q} \|f\|_{\dot{M}^1_q}$$

Thus, for $p > 1$, we have $\|\chi_{B(x,r)} M_f\|_p \leq C(\|f_1\|_p + r^{-d/q} \|\chi_{B(x,r)}\|_p \|f\|_{\dot{M}^1_q})$ and $\|\chi_{B(x,r)} M_f\|_{L^{p,*}} \leq C(\|f_1\|_{L^{p,*}} + r^{-d/q} \|\chi_{B(x,r)}\|_{L^{p,*}} \|f\|_{\dot{M}^1_q})$, while, for $f = d\mu$,

$$\begin{aligned} \mathcal{N}(\chi_{B(x,r)} M_f) &\leq 2(\mathcal{N}(\chi_{B(x,r)} M_{f_1}) + \mathcal{N}(\chi_{B(x,r)} M_{f_2})) \\ &\leq C(\int_{B(x,r)} d|\mu| + r^{-d/q} \|\chi_{B(x,r)}\|_1 \|f\|_{\dot{M}^1_q}). \end{aligned}$$

Thus, Proposition 3 is proved.

6 Riesz potentials and Morrey spaces

The next tool we will discuss is the Riesz potentials of a function (or a measure) in a Morrey space. We begin with a variant of Lemma 1.

Lemma 3
Let $0 \leq \gamma < d$. If ω is a non-negative radially decreasing function on $\mathbb{R}^d$ and $f = d\mu$ a locally finite measure, then

$$|\int_{\mathbb{R}^d} \omega(x-y) d\mu(y)| \leq C \|\omega(x)|x|^{-\gamma}\|_1 \sup_{r>0} \frac{1}{r^{d-\gamma}} \int_{|x-y|<r} d|\mu|(y) \tag{16}$$

Indeed, let us define $M_\gamma(d\mu)$ as

$$M_\gamma(d\mu)(x) = \sup_{r>0} \frac{1}{r^{d-\gamma}} \int_{|x-y|<r} d|\mu|(y) \tag{17}$$

Writing $\omega(x) = \theta(|x|)$, we find :

$$\begin{aligned}
|\int_{\mathbb{R}^d} \omega(x-y)d\mu(y)| &\leq \sum_{j\in\mathbb{Z}} \theta(2^j) \int_{2^j<|x-y|\leq 2^{j+1}} d|\mu|(y) \\
&\leq \sum_{j\in\mathbb{Z}} (\theta(2^{j-1}) - \theta(2^j)) 2^{j(d-\gamma) M_\gamma(d\mu)(x)} \\
&= \sum_{j\in\mathbb{Z}} \theta(2^j)(2^{(j+1)(d-\gamma)} - 2^{j(d-\gamma)}) M_\gamma(d\mu)(x) \\
&= c_\gamma \sum_{j\in\mathbb{Z}} \theta(2^j) \int_{2^{j-1}<|x-y|\leq 2^j} \frac{dy}{|y|^\gamma} M_\gamma(d\mu)(x) \\
&\leq c_\gamma \int \omega(y) \frac{dy}{|y|^\gamma} M_\gamma(d\mu)(x).
\end{aligned}$$

Thus, Lemma 3 is proved. The same proof works for a truncated kernel : if we integrate only for $|x-y| > A$, we find that

$$|\int_{|x-y|>A} \omega(x-y)d\mu(y)| \leq c_\gamma \int_{|y|>A/4} \frac{dy}{|y|^\gamma}\ M_\gamma(d\mu)(x). \tag{18}$$

A direct consequence of this lemma is the Adams-Hedberg inequality for Riesz potentials [7, 1] :

Lemma 4 (Adams–Hedberg inequality)
Let $0 < r < \gamma < d$. Then we have

$$|I_r(d\mu)(x)| \leq c_{\gamma,r} M_{d\mu}(x)^{(\gamma-r)/\gamma} M_\gamma(d\mu)(x)^{r/\gamma}. \tag{19}$$

To prove the inequality, it is enough to split the integration defining $I_r(d\mu)$ into $|x-y| < A$ and $|x-y| \leq A$, and we get from Lemmas 1 and 3 :

$$\begin{aligned}
|I_r(d\mu)(x)| &\leq C(M_{d\mu}(x) \int_{|y|<A} \frac{dy}{|y|^{d-r}} + M_\gamma(d\mu)(x) \int_{|y|>A/4} \frac{dy}{|y|^{d-r+\gamma}}) \\
&\leq C'(A^r M_{d\mu}(x) + A^{r-\gamma} M_\gamma(d\mu)(x))
\end{aligned}$$

and we conclude by taking $A = \left(\frac{M_\gamma(d\mu)(x))}{M_{d\mu}(x)}\right)^{1/\gamma}$.

Proposition 4 (Riesz potentials and Morrey spaces)
Let $0 < r < d/q$. Let $\lambda = 1 - \frac{rq}{d}$. Then :
A) if $1 < p \leq q$, I_r maps $\dot{M}^p_q$ to $\dot{M}^{p/\lambda}_{q/\lambda}$ and $\dot{M}^{p,*}_q$ to $\dot{M}^{p/\lambda,*}_{q/\lambda}$:

$$\|I_r f\|_{\dot{M}^{p/\lambda}_{q/\lambda}} \leq c_{p,q,r} \|f\|_{\dot{M}^p_q} \text{ and } \|I_r f\|_{\dot{M}^{p/\lambda,*}_{q/\lambda}} \leq c_{p,q,r} \|f\|_{\dot{M}^{p,*}_q} \tag{20}$$

B) if $1 = p \leq q$, I_r *maps* $\dot M^p_q$ *to* $\dot M^{p/\lambda,*}_{q/\lambda}$ *:*

$$\|I_r f\|_{\dot M^{1/\lambda}_{q/\lambda}} \leq c_{q,r} \|f\|_{\dot M^1_q} \tag{21}$$

Proposition 4 is a consequence of the Adams-Hedberg inequality : we have $\dot M^p_q \subset M^1_q$ and $M^{p,*}_q \subset \dot M^1_q$; for $d\mu \in \dot M^1_q$, we have

$$\sup_{x\in\mathbb{R}^d} \mathcal{M}_{d/q}(d\mu)(x) \leq \|d\mu\|_{\dot M^1_q}$$

and thus

$$I_r(d\mu) \leq C_{r,q} \mathcal{M}_{d\mu}(x)^\lambda \|d\mu\|^{1-\lambda}_{\dot M^1_q}.$$

We conclude by using Proposition 3.

7 Estimates for the bilinear operators.

We are going to prove Part A) of Theorem 1. We begin with the classical following lemma :

Lemma 5
Let $0 < \beta < 1$. *The maximal function* M_β *of the function* $|t|^{-\beta/2}$ *satisfies* $M_\beta(t) \leq C_\beta |t|^{-\beta/2}$.

The proof is easy : if $r \leq |t|/2$, $\frac{1}{2r}\int_{|s-t|<r} \frac{ds}{|s|^{\beta/2}} \leq \frac{2^{\beta/2}}{t^{\beta/2}}$; if $r > |t|/2$, $\frac{1}{2r}\int_{|s-t|<r} \frac{ds}{|s|^{\beta/2}} \leq \frac{1}{2r}\int_{|s|<3r} \frac{ds}{|s|^{\beta/2}} = \frac{2\ 3^{1-\beta/2}}{(1-\beta/2)r^{\beta/2}} \leq \frac{2\ 2^{\beta/2}\ 3^{1-\beta/2}}{(1-\beta/2)} \frac{1}{|t|^{\beta/2}}$.

When f and h belong to E_β, we may write $|f(t,x)| \leq t^{-\beta/2} F(x)$ and $|h(t,x)| \leq t^{-\beta/2} H(x)$ where $F, H \in \dot M^{2/(2-\beta),*}_{2/(2-\beta)}$. Using Proposition 1, we find that

$$\begin{aligned} |B_{\alpha,\epsilon}(f,h)| &\leq C \int_0^t \frac{1}{(\sqrt{t-s} + |x-y|)^{d+1}} s^{-\beta} F(y) I_1 H(y)\ ds\ dy \\ &\leq C \int_0^t \frac{1}{(\sqrt{t-s})^{2-\beta}} s^{-\beta}\ ds \int \frac{1}{|x-y|^{d+\beta-1}} F(y) I_1 H(y)\ dy \end{aligned}$$

Since

$$\int_0^t (t-s)^{-1+\frac{\beta}{2}} s^{-\beta}\ ds = \gamma_\beta t^{-\beta/2}$$

we find that

$$|B_{\alpha,\epsilon}(f,h)| \leq C_\beta t^{-\beta/2} I_{1-\beta}(F\, I_1 H)(x)$$

where the constant C_β does not depend on α nor on ϵ.

From Proposition 4, we get that for $H \in \dot{M}^{2/(2-\beta),*}_{d/(2-\beta)}$, we have $I_1(H) \in \dot{M}^{2/(\gamma(2-\beta)),*}_{d/(\gamma(2-\beta))}$ with $\gamma = 1 - \frac{1}{2-\beta}$. Thus, we get that $F\ I_1(H) \in \dot{M}^{p_0,*}_{q_0}$ with $\frac{1}{q_0} = \frac{1}{d}(2-\beta)(1+\gamma)$ and $\frac{1}{p_0} = \frac{1}{2}(2-\beta)(1+\gamma) = \frac{3-2\beta}{2}$. Moreover, since $1/2 < \beta < 1$, we have

$$1 - \beta < 3 - 2\beta = \frac{d}{q_0} \text{ and } 1 < \frac{2}{3-2\beta} = p_0.$$

Thus, we may apply again Proposition 4 and find that $I_{1-\beta}(FI_1(H)) \in \dot{M}^{p_0/\delta,*}_{q_0/\delta}$ with $\delta = 1 - \frac{q_0(1-\beta)}{d}$. But we have

$$\frac{\delta}{q_0} = \frac{1}{q_0} - \frac{1-\beta}{d} = \frac{2-\beta}{d} \text{ and } \frac{\delta}{p_0} = \frac{1}{p_0} - \frac{1-\beta}{d}\frac{q_0}{p_0} = \frac{2-\beta}{2}.$$

Thus, Theorem 1, Part A) is proved.

8 Parabolic-elliptic and parabolic-parabolic Keller–Segel equations

The proof of Theorem 1, Part B), is now easy. We want to solve

$$u_{\alpha,\epsilon} = W_t * u_0 - \int_0^t \text{div } W_{t-s} * (u_{\alpha,\epsilon}\ L_{\alpha,\epsilon} u_{\alpha,\epsilon})\ ds.$$

This is done with the contraction principle.

First, we check that for $1/2 < \beta < 1$, we have $W_t * u_0 \in E_\beta$ where

$$E_\beta = \{u(t,x)\ /\ \|u\|_{E_\beta} = \sup_{t>0} t^{\beta/2} u(t,x) \in \dot{M}^{2/(2-\beta),*}_{d/(2-\beta)}\}.$$

Indeed, we just write $W_t(x) \le C\frac{1}{(\sqrt{t}+|x|)^d}$, hence $t^{\beta/2}W_t(x) \le C\frac{1}{|x|^{d-\beta}}$, so that $|t^{\beta/2} W_t * u_0(x)| \le C I_\beta(|u_0|)(x)$. We then use Proposition 4.

Then, one easily concludes from the following classical lemma (see [13] for instance) :

Lemma 6 (Picard iterates for bilinear operators)
Let B is a bounded bilinear operator on a Banach space E :

$$\|B(x,y)\|_E \le C_B \|x\|_E \|y\|_E.$$

Let $x_0 \in E$ be such that $\|x_0\|_E < \frac{1}{4C_B}$. Then the iterates $x_{n+1} = x_0 - B(x_n, x_n)$ converge in the E norm to a solution of the equation

$$x = x_0 - B(x,x)$$

such that $\|x\|_E \le 2\|x_0\|_E$. This solution is unique in the ball $B(0, \frac{1}{2C_B})$.

9 The convergence from the parabolic-parabolic equations to the parabolic-elliptic equations

For the proof of Theorem 1, Part C), we shall follow the strategy of [2].

When f and h belong to E_β, we may write $|f(t,x)| \leq t^{-\beta/2} F(x)$ and $|h(t,x)| \leq t^{-\beta/2} H(x)$ where $F, H \in \dot{M}^{2/(2-\beta),*}_{2/(2-\beta)}$. The key estimattes for Part A) were then :

$$|\int_0^t \text{ div } W_{t-s} * (f L_{\alpha,\epsilon} h) \, ds| \leq C_0 t^{-\beta/2} I_{1-\beta}(F\, I_1 H)(x)$$

and

$$\|I_{1-\beta}(F\, I_1 H)\|_{\dot{M}^{2/(2-\beta),*}_{2/(2-\beta)}} \leq C_1 \|F\|_{\dot{M}^{2/(2-\beta),*}_{2/(2-\beta)}} \|H\|_{\dot{M}^{2/(2-\beta),*}_{2/(2-\beta)}}$$

where the constants C_0 and C_1 don't depend on α nor on ϵ.

When we want to compare $u_{\alpha,\epsilon}$ and $u_{\alpha,0}$, we write :

$$\begin{aligned} u_{\alpha,\epsilon} - u_{\alpha,0} = & \int_0^t \text{ div } W_{t-s} * ((u_{\alpha,0} - u_{\alpha,\epsilon}) L_{\alpha,\epsilon} u_{\alpha,\epsilon}) \, ds \\ & + \int_0^t \text{ div } W_{t-s} * (u_{\alpha,0} L_{\alpha,\epsilon}(u_{\alpha,0} - u_{\alpha,\epsilon})) \, ds \\ & + \int_0^t \text{ div } W_{t-s} * (u_{\alpha,0}(L_{\alpha,0} - L_{\alpha,\epsilon}) u_{\alpha,0}) \, ds, \end{aligned}$$

hence

$$\begin{aligned} \|u_{\alpha,\epsilon} - u_{\alpha,0}\|_{E_\beta} \leq & C_0 C_1 \|u_{\alpha,0} - u_{\alpha,\epsilon}\|_{E_\beta} (\|u_{\alpha,\epsilon}\|_{E_\beta} + \|u_{\alpha,0}\|_{E_\beta}) \\ & + \| \int_0^t \text{ div } W_{t-s} * (u_{\alpha,0}(L_{\alpha,0} - L_{\alpha,\epsilon}) u_{\alpha,0}) \, ds \|_{E_\beta}. \end{aligned}$$

If $\|u_0\|_{\dot{M}^1_{d/2}}$ is small enough (so that $\|W_t * u_0\|_{E_\beta} \leq \delta_0 < \frac{1}{4C_0C_1}$), the solutions $u_{\alpha,\epsilon}$ and $u_{\alpha,0}$ will be small in E_β ($\|u_{\alpha,\epsilon}\|_{E_\beta} \leq 2\delta_0$ and $\|u_{\alpha,0}\|_{E_\beta} \leq 2\delta_0$) and we find

$$\|u_{\alpha,\epsilon} - u_{\alpha,0}\|_{E_\beta} \leq \frac{1}{1 - 4C_0C_1\delta_0} \| \int_0^t \text{ div } W_{t-s} * (u_{\alpha,0}(L_{\alpha,0} - L_{\alpha,\epsilon}) u_{\alpha,0}) \, ds \|_{E_\beta}.$$

Thus, we are lead to prove that

$$\lim_{\epsilon \to 0} \| \int_0^t \text{ div } W_{t-s} * (u_{\alpha,0}(L_{\alpha,0} - L_{\alpha,\epsilon}) u_{\alpha,0}) \, ds \|_{E_\beta} = 0.$$

Let $F_{\beta,\gamma}$ be defined, for $0 \leq \gamma < 1-\beta$, as

$$F_{\beta,\gamma} = \{u(t,x) \ / \ \sup_{t>0} t^{(\beta+\gamma)/2} u(t,x) \in \dot{M}^{2/(1-(\beta+\gamma)),*}_{d/(1-(\beta+\gamma))}\}.$$

The space $F_{\beta,\gamma}$ is normed with

$$\|u\|_{F_{\beta,\gamma}} = \|\sup_{t>0} t^{(\beta+\gamma)/2} u(t,x)\|_{\dot{M}^{2/(1-(\beta+\gamma)),*}_{d/(1-(\beta+\gamma))}}.$$

If $f \in E_\beta$ ($|f(t,x)| \leq t^{-\beta/2} F(x)$ with $F \in \dot{M}^{2/(2-\beta),*}_{d/(2-\beta)}$) and $k \in F_{\beta,\gamma}$ ($|k(t,x)| \leq t^{-(\beta+\gamma)/2} K(x)$ with $K \in \dot{M}^{1/(1-(\beta+\gamma)),*}_{d/(1-(\beta+\gamma))}$), then we write

$$|\text{div } W_{t-s}(x-y)| \leq C t^{-1+(\beta+\gamma)/2} \frac{1}{|x-y|^{d-1+\beta+\gamma}}$$

and thus

$$|\int_0^t \text{div } W_{t-s} * (fk) \ ds| \leq C t^{-\beta/2} I_{1-\gamma-\beta}(FK)(x).$$

We have $FK \in \dot{M}^{2/(3-2\beta-\gamma),*}_{d/(3-2\beta-\gamma)}$, hence $I_{1-\gamma-\beta}(FK) \in \dot{M}^{2/(2-\beta),*}_{d/(2-\beta)}$. Hence, we get, for $0 < \gamma < 1-\beta$, that :

$$\| \int_0^t \text{div } W_{t-s} * (u_{\alpha,0}(L_{\alpha,0} - L_{\alpha,\epsilon}) u_{\alpha,0}) \ ds\|_{E_\beta} \leq C \|u_{\alpha,0}\|_{E_\beta} \|(L_{\alpha,0} - L_{\alpha,\epsilon}) u_{\alpha,0}\|_{F_{\beta,0}+F_{\beta,\gamma}}.$$

Thus, we are lead to prove that we may write $(L_{\alpha,0} - L_{\alpha,\epsilon}) u_{\alpha,0} = v_{\alpha,\epsilon} + w_{\alpha,\epsilon}$ with

$$\lim_{\epsilon \to 0} \|v_{\alpha,\epsilon}\|_{F_{\beta,0}} + \|w_{\alpha,\epsilon}\|_{F_{\beta,\gamma}} = 0.$$

Let $\eta = \epsilon^\delta$, with $0 < \delta < 1$. we may assume that $\eta < 1/2$ (since we are interested in $\epsilon \to 0$). Let $q_{\alpha,\epsilon}(t,x) = L_{\alpha,\epsilon} u_{\alpha,0}(t,x)$ and $q_{\alpha,0}(t,x) = L_{\alpha,0} u_{\alpha,0}(t,x)$. We define

$$w_{\alpha,\epsilon} = \frac{1}{\epsilon} e^{-\alpha \frac{\eta t}{\epsilon}} W_{\frac{\eta t}{\epsilon}} * (q_{\alpha,\epsilon}((1-\eta)t, x) - q_{\alpha,0}(t,x))$$

We know that $q_{\alpha,\epsilon}$ and $q_{\alpha,0}$ belong to $F_{\beta,0}$ (uniformly with respect to α and ϵ), and so does $q_{\alpha,\epsilon}((1-\eta)t, x)$. We then write that

$$e^{-\alpha \frac{\eta t}{\epsilon}} W_{\frac{\eta t}{\epsilon}}(x-y) \leq C (\frac{\eta t}{\epsilon})^{-\gamma/2} \frac{1}{|x-y|^{d-\gamma}}.$$

Thus, we obtain

$$\|w_{\alpha,\epsilon}\|_{F_{\beta,\gamma}} \leq C_{u_0} \epsilon^{\frac{\gamma(1-\delta)}{2}}$$

where the constant C_{u_0} depends on u_0, but not on α nor on ϵ.

Let $Q_{\alpha,\epsilon,t}(s,x) = q_{\alpha,\epsilon}(s,x)$ and $Q_{\alpha,0,t}(s,x) = q_{\alpha,0}(t,x)$. We have

$$\epsilon\partial_s Q_{\alpha,\epsilon,t}(s,x) = \Delta Q_{\alpha,\epsilon,t}(s,x) - \alpha Q_{\alpha,\epsilon,t}(s,x) + \vec{\nabla} u_{\alpha,0}(s,x)$$

with

$$Q_{\alpha,\epsilon,t}((1-\eta)t,x) = q_{\alpha,\epsilon}((1-\eta)t,x)$$

while

$$0 = \epsilon\partial_s Q_{\alpha,0,t}(s,x) = \Delta Q_{\alpha,0,t}(s,x) - \alpha Q_{\alpha,0,t}(s,x) + \vec{\nabla} u_{\alpha,0}(t,x)$$

with

$$Q_{\alpha,0,t}((1-\eta)t,x) = q_{\alpha,0}(t,x)$$

We have the identity

$$\begin{aligned} v_{\alpha,\epsilon}(t) =& Q_{\alpha,0,t}(t,x) - \frac{1}{\epsilon} e^{-\alpha\frac{\eta t}{\epsilon}} W_{\frac{\eta t}{\epsilon}} * Q_{\alpha,0,t}((1-\eta)t,x) \\ & - Q_{\alpha,\epsilon,t}(t,x) + \frac{1}{\epsilon} e^{-\alpha\frac{\eta t}{\epsilon}} W_{\frac{\eta t}{\epsilon}} * Q_{\alpha,\epsilon,t}((1-\eta)t,x) \end{aligned}$$

or, equivalently,

$$v_{\alpha,\epsilon} = \frac{1}{\epsilon}\int_{(1-\eta)t}^{t} \vec{\nabla} W_{\frac{t-s}{\epsilon}} * (u_{\alpha,0}(s,x) - u_{\alpha,0}(t,x))\, ds.$$

If $U_\eta(s,x) = \sup_{(1-\eta)t<s<t} |u_{\alpha,0}(s,x) - u_{\alpha,0}(t,x)|$, we find that

$$\|v_{\alpha,\epsilon}\|_{F_{\beta,0}} \leq C\|U_\eta\|_{E_\beta}.$$

It remains to estimate U_η. We know that $u_{\alpha,0}$ belongs to E_β, hence $|u_{\alpha,0}(t,x)| \leq t^{-\beta/2} V_\alpha(x)$ where $V_\alpha \in \dot{M}^{2/(2-\beta),*}_{d/(2-\beta)}$. We write, for $(1-\eta)t < s < t$, $u_{\alpha,0}(t,x) - u_{\alpha,0}(s,x) = A_\alpha(t,s,x) - B_\alpha(t,s,x) - C_\alpha(t,s,x)$ with

$$\begin{cases} A_\alpha(t,s,x) = & (W_t - W_s) * u_0 \\ B_\alpha(t,s,x) = & \int_0^s \text{div } (W_{t-\sigma} - W_{s-\sigma}) * (u_{\alpha,0}\, L_{\alpha,0} u_{\alpha,0})\, d\sigma \\ C_\alpha(t,s,x) = & \int_s^t \text{div } W_{t-\sigma} * (u_{\alpha,0}\, L_{\alpha,0} u_{\alpha,0})\, d\sigma \end{cases}$$

The control of A_α is easy :

$$\begin{aligned}|A_\alpha(t,s,x)| &\leq \int |W_t(x-y)-W_s(t-y)||u_0(y)|\ dy\\ &\leq (t-s)\int \sup_{s<\sigma<t}|\partial_\sigma W_\sigma(x-y)||u_0(y)|\ dy\\ &\leq C|t-s|\int \frac{1}{(\sqrt{t}+|x-y|)^{d+2}}|u_0(y)|\ dy\\ &\leq C'\frac{t-s}{t^{1+\frac{\beta}{2}}}I_\beta|u_0|(x)\end{aligned}$$

Thus, we find that $\|\sup_{(1-\eta)t<s<t}|A_\alpha(t,s,x)|\|_{E_\beta} \leq C\eta\|I_\beta(|u_0|)\|_{\dot{M}^{2/(2-\beta),*}_{d/(2-\beta)}} = O(\eta)$.

The control of C_α is easy as well. We have

$$\begin{aligned}|C_\alpha(t,s,x)| &\leq C\int_s^t \frac{1}{(\sqrt{t-\sigma}+|x-y|)^{d+1}}\sigma^{-\beta}V_\alpha(y)I_1V_\alpha(y)\ dy\ d\sigma\\ &\leq C't^{-\beta}\int_s^t \frac{1}{(t-\sigma)^{1-\frac{\beta}{2}}}\ d\sigma \quad I_{1-\beta}(V_\alpha\, I_1V_\alpha)(x)\\ &= C''\frac{(t-s)^{\beta/2}}{t^\beta}I_{1-\beta}(V_\alpha\, I_1V_\alpha)(x).\end{aligned}$$

Thus, we find that $\|\sup_{(1-\eta)t<s<t}|C_\alpha(t,s,x)|\|_{E_\beta} \leq C\eta^{\beta/2}\|I_{1-\beta}(V_\alpha\, I_1V_\alpha))\|_{\dot{M}^{2/(2-\beta),*}_{d/(2-\beta)}} = O(\eta^{\beta/2})$.

The control of B_α is a bit trickier. We write

$$D_\alpha(s,x)=\int_0^s \text{div}\ W_{s-\sigma}*(u_{\alpha,0}\, L_{\alpha,0}u_{\alpha,0})\ d\sigma$$

so that

$$B_\alpha(t,s,x)=W_{t-s}*D_\alpha(s,x)-D_\alpha(s,x)=\int_0^{(t-s)}\Delta W_\tau * D_\alpha(s,x)\ d\tau$$

and finally, since $\Delta W_\tau * D_\alpha(s,x) = (-\Delta)^{1-\theta}W_\tau*(-\Delta)^\theta D_\alpha(s,x)$ (where $0<\theta<1-\beta<\beta$),

$$|B_\alpha(t,s,x)|\leq C\int_0^{t-s}\int_0^s\iint \frac{1}{(\sqrt{\tau}+|x-y|)^{d+2-2\theta}}\frac{V_\alpha(z)I_1V_\alpha(z)}{(\sqrt{s-\sigma}+|y-z|)^{d+1+2\theta}\sigma^\beta}\,dz\,dy\,d\sigma\,d\tau$$

We then write

$$|B_\alpha(t,s,x)|\leq C\int_0^{t-s}\int_0^s\iint \frac{1}{\tau^{1-\theta/2}|x-y|^{d-\theta}}\frac{V_\alpha(z)I_1V_\alpha(z)}{(s-\sigma)^{1+(\theta-\beta)/2}|y-z|^{d+\beta+\theta-1}\sigma^\beta}\,dz\,dy\,d\sigma\,d\tau$$

which gives

$$|B_\alpha(t,s,x)| \leq C \int_0^{t-s} \int_0^s \frac{1}{\tau^{1-\theta/2}} \frac{1}{(s-\sigma)^{1+(\theta-\beta)/2}\sigma^\beta} \, d\sigma \, d\tau \quad I_\theta(I_{1-\beta-\theta}(V_\alpha \, I_1 V_\alpha)(x)$$

and finally

$$|B_\alpha(t,s,x)| \leq C \frac{(t-s)^{\theta/2}}{s^{(\beta+\theta)/2}} I_{1-\beta}(V_\alpha \, I_1 V_\alpha)(x).$$

Thus, we find that $\| \sup_{(1-\eta)t<s<t} |B_\alpha(t,s,x)| \|_{E_\beta} \leq C\eta^{\theta/2} \| I_{1-\beta}(V_\alpha \, I_1 V_\alpha)) \|_{\dot{M}^{2/(2-\beta),*}_{d/(2-\beta)}} = O(\eta^{\theta/2})$.

We have thus proved that $\|u_{\alpha,\epsilon} - u_{\alpha,0}\|_{E_\beta} = O(\epsilon^\kappa)$ with $\kappa = \min(\frac{\gamma(1-\delta)}{2}, \frac{\theta\delta}{2}) > 0$. Theorem 1 is proved.

10 Fractional diffusion

The Keller-Segel equations have been generalized with the introduction of non-local diffusion [4]. The equations are then the following :

- the parabolic-elliptic model :

$$\begin{cases} \partial_t u + (-\Delta)^{\theta/2} u = -\operatorname{div}(u \vec{\nabla} \varphi) \\ (-\Delta)^{\theta/2} \varphi = -\alpha \varphi + u \\ u = u_0 \text{ for } t = 0 \end{cases}$$

- the parabolic-parabolic model :

$$\begin{cases} \partial_t u + (-\Delta)^{\theta/2} u = -\operatorname{div}(u \vec{\nabla} \varphi) \\ \epsilon \partial_t \varphi + (-\Delta)^{\theta/2} \varphi = -\alpha \varphi + u \\ u = u_0 \text{ for } t = 0 \\ \varphi = 0 \text{ for } t = 0 \end{cases}$$

where $\theta \in (1,2]$.

Let $\mathcal{W}_\theta$ be the function whose Fourier transform is $e^{-|\xi|^\theta}$ and $\mathcal{W}_{t,\theta}(x) = \frac{1}{t^{d/\theta}} \mathcal{W}_\theta(\frac{x}{t^{1/\theta}})$. We are lead to solve the integral equation

$$u = \mathcal{W}_{t,\theta} * u_0 - \int_0^t \operatorname{div} \mathcal{W}_{t-s,\theta} * (u(L_{\alpha,\epsilon,\theta} u)) \, ds$$

where $L_{\alpha,\epsilon,\theta}$ is the operator

$$L_{\alpha,\epsilon,\theta} u(t,x) = \frac{1}{\epsilon} \int_0^t e^{-\alpha \frac{t-\sigma}{\epsilon}} \vec{\nabla} W_{\frac{t-\sigma}{\epsilon},\theta} * u \, d\sigma \text{ if } \alpha \geq 0 \text{ and } \epsilon > 0,$$

$$L_{\alpha,\epsilon,\theta}u(t,x) = u * \vec{\nabla} J_{\alpha,\theta} \text{ if } \epsilon = 0 \text{ and } \alpha > 0,$$

$$L_{\alpha,\epsilon,\theta}u(t,x) = u * \vec{\nabla} G_\theta \text{ if } \epsilon = \alpha = 0.$$

where $G_\theta * u = (-\Delta)^{-\theta/2}u$ and $J_{\alpha,\theta} = \int_0^\infty W_{s,\theta}e^{-\alpha s}\, ds$.

We have $|\mathcal{W}_{t,\theta}| \leq C\frac{1}{t^{(1/\theta}+|x|)^d}$ and $|\partial_j \mathcal{W}_{t,\theta}| \leq C\frac{1}{t^{(1/\theta}+|x|)^{d+1}}$. We thus find that :

$$|\int_0^t \text{div } \mathcal{W}_{t-s,\theta} * (u(\frac{1}{\epsilon}\int_0^s \vec{\nabla}\mathcal{W}_{\frac{s-\sigma}{\epsilon},\theta} * v\, d\sigma))\, ds|$$
$$\leq C_0 \int_0^t \int \frac{1}{((t-s)^{1/\theta}+|x-y|)^{d+1}} |u(s,y)| I_{\theta-1}(M_*|v|)(s,y)\, dy\, ds$$

Then it is easy to see that the optimal space to pick the initial value for those Keller-Segel equations is the space $\dot{M}^1_{\frac{d}{2(\theta-1)}}$, and that the space where to solve them is

$$E_{\beta,\theta} = \{u(t,x) \ / \ \sup_{t>0} t^{\beta/\theta} u(t,x) \in \dot{M}^{(2\theta-2)/(2\theta-2-\beta),*}_{d/(2\theta-2-\beta)}\}$$

where $\frac{\theta-1}{2} < \beta < \theta - 1$.

Then one may adapt Theorem 1 into :

Theorem 2 (Fractional Keller-Segel equations)
Let $\theta \in (1,2]$ and $\frac{\theta-1}{2} < \beta < \theta - 1$.
A) There exists a positive $\delta_0 > 0$ such that, for every $u_0 \in \dot{M}^1_{\frac{d}{2(\theta-1)}}(\mathbb{R}^d)$ with $\|u_0\|_{\dot{M}^1_{\frac{d}{2(\theta-1)}}((\mathbb{R}^d)} < \delta_0$, for every $\alpha \geq 0$ and every $\epsilon \geq 0$, the Picard iterates

$$v_{\alpha,\epsilon,0} = W_{t,\theta} * u_0 \text{ and } v_{\alpha,\epsilon,n+1} = v_0 - \int_0^t \text{div } W_{t-s,\theta} * (v_{\alpha,\epsilon,n} L_{\alpha,\epsilon,\theta}(v_{\alpha,\epsilon,n}))\, ds$$

converge in the $E_{\beta,\theta}$ norm to a solution of the Keller-Segel equation $u_{\alpha,\epsilon} = W_{t,\theta} * u_0 - \int_0^t \text{div } W_{t-s,\theta} * (u_{\alpha,\epsilon}\ L_{\alpha,\epsilon,\theta}u_{\alpha,\epsilon})\, ds$.
B) Moreover, when ϵ goes to 0, the solution $u_{\alpha,\epsilon}$ of the parabolic-parabolic problem converges in the $E_{\beta,\theta}$ norm to the solution $u_{\alpha,0}$ of the parabolic-elliptic problem.

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