Sobolev multipliers, maximal functions and parabolic equations with a quadratic nonlinearity.

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Abstract

We develop a general framework to describe global mild solutions for a Cauchy problem with small initial values concerning a general class of semilinear parabolic equations with quadratic nonlinearity. This class includes the Navier–Sokes equations, the subcritical dissipative quasi-geostrophic equation and the parabolic–elliptic Keller-Segel system.

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Introduction.

In this paper, we shall study parabolic semi-linear equations on $(0,+\infty)\times\mathbb{R}^n$ from the type :

$$\partial_t u + (-\Delta)^{\alpha/2} u = (-\Delta)^{\beta/2} u^2 \tag{1}$$

with $0<\alpha<n+2\beta$ and $0<\beta<\alpha$.

More generally, we consider the following Cauchy problem : given $\vec u_0 \in (\mathcal{S}'(\mathbb{R}^n))^d$, find a vector distribution $\vec u$ on $(0,+\infty)\times\mathbb{R}^n$ (or on $(0,T)\times\mathbb{R}^n$) so that, for $i=1,\dots,d$ we have

$$\partial_t u_i = -(-\Delta)^{\alpha/2} u_i + \sum_{j=1}^d \sum_{k=1}^d \sigma_{i,j,k}(D)(u_j u_k) \tag{2}$$

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and

$$\lim_{t\to 0} u_i(t,x) = u_{i,0}. \tag{3}$$

We assume that $\sigma_{i,j,k}(D)$ is an homogeneous pseudo-differential operator of degree β with $0 < \beta < \alpha < n+2\beta$: for $f \in \mathcal{S}(\mathbb{R}^n)$ with Fourier transform $\mathcal{F}f$, we have :

$$\sigma_{i,j,k}(D)f = \mathcal{F}^{-1}\big(\sigma_{i,j,k}(\xi)\mathcal{F}f(\xi)\big) \tag{4}$$

where $\sigma_{i,j,k}$ is a (positively) smooth homogeneous function of degree β on $\mathbb{R}^n - \{0\}$:

$$\text{for } \lambda > 0 \text{ and } \xi \neq 0, \quad \sigma_{i,,j,k}(\lambda\xi) = \lambda^\beta \sigma_{i,j,k}(\xi) \tag{5}$$

We rewrite equation (2) in a vectorial form :

$$\partial_t \vec{u} = -(-\Delta)^{\alpha/2}\vec{u} + \sigma(D)(\vec{u}\otimes\vec{u}) \tag{6}$$

and use Duhamel's formula to turn the problem into an integral problem :

$$\vec{u} = e^{-t(-\Delta)^{\alpha/2}}\vec{u}_0 + \int_0^t e^{-(t-s)(-\Delta)^{\alpha/2}}\sigma(D)(\vec{u}\otimes\vec{u})\, ds. \tag{7}$$

We shall use the classical estimate :

Lemma 1 *There exists a constant C_0 (depending on σ) such that, for two functions $\vec{f}$ and $\vec{v}$ on $\mathbb{R}^n$ (with values in $\mathbb{R}^d$) we have*

$$|e^{-t(-\Delta)^{\alpha/2}}\sigma(D)(\vec{u}\otimes\vec{v})| \leq C_0 \int_{\mathbb{R}^n} \frac{|\vec{u}(y)||\vec{v}(y)|}{(|t-s|^{1/\alpha}+|x-y|)^{n+\beta}}\, dy \tag{8}$$

This lemma will be proved in Section 1.

The core of the paper is the discussion of the equation

$$U(t,x) = U_0(t,x) + \iint_{\mathbb{R}\times\mathbb{R}^n} K_{\alpha,\beta}(t-s,x-y)\, U^2(s,y)\, ds\, dy \tag{9}$$

with

$$K_{\alpha,\beta}(t,x) = C_0 \frac{1}{(|t|^{1/\alpha}+|x|)^{n+\beta}} \tag{10}$$

and $U_0 \geq 0$.

1 Proof of Lemma 1.

Due to homogeneity, it is enough to prove that :

$$|\int_{\mathbb{R}^n} e^{ix.\xi} e^{-|\xi|^\alpha} \sigma(\xi)\ d\xi| \leq (2\pi)^n C_0 \frac{1}{(1+|x|)^{n+\beta}} \tag{11}$$

Let $\theta \in \mathcal{D}(\mathbb{R}^n)$ with $1_{B(0,1)} \leq \theta \leq 1_{B(0,2)}$ and let $N > \frac{n+\beta}{2}$. Let

$$I(x) = \int_{\mathbb{R}^n} e^{ix.\xi} e^{-|\xi|^\alpha} \sigma(\xi)\ d\xi. \tag{12}$$

and

$$J(x) = \int_{\mathbb{R}^n} e^{ix.\xi} e^{-|\xi|^\alpha} \sigma(\xi)\theta(\xi)\ d\xi. \tag{13}$$

$I - J$ is a smooth function with rapid decay (it belongs to $\mathcal{S}$). Of course, J is a bounded function, since its Fourier transform is integrable. Thus, we consider only the case $|x| > 1$. For $R > 0$, we write

$$J(x) = \begin{cases} \displaystyle\int_{\mathbb{R}^n} e^{ix.\xi} e^{-|\xi|^\alpha} \sigma(\xi)\theta(\frac{\xi}{R})\theta(x)\ d\xi + \\ \displaystyle\int_{\mathbb{R}^n} \frac{e^{ix.\xi}}{(-|x|^2)^N} \Delta_\xi^N \left(e^{-|\xi|^\alpha} \sigma(\xi)\theta(\xi)(1-\theta(\frac{\xi}{R}))\right)\ d\xi \end{cases} \tag{14}$$

which gives, for a constant C_N which depend on σ, on θ and on N (but neither on x nor on R) :

$$|J(x)| \leq \int_{|\xi|<2R} C_N |\xi|^\beta d\xi + \int_{|\xi|>R} \frac{1}{(|x|^2)^N} C_N |\xi|^{\beta-2N}\ d\xi \tag{15}$$

Thus

$$|J(x)| \leq C(R^{n+\beta} + \frac{R^{n+\beta-2N}}{|x|^{2N}}) \tag{16}$$

and we conclude the proof of Lemma 1 by taking $R = \frac{1}{|x|}$.

2 Semilinear equation with a positive kernel.

In this section, before discussing equation (9), we discuss the general integral equation

$$f(x) = f_0(x) + \int_X K(x,y) f^2(y)\ d\mu(y) \tag{17}$$

where μ is a non-negative σ-finite measure on a space X ($X = \cup_{n\mathbb{N}} Y_n$ with $\mu(Y_n) < +\infty$), and K is a positive measurable function on $X \times X$: $K(x,y) > 0$ almost everywhere. We shall make a stronger assumption on K : there exists a sequence X_n of measurable subsets of X such that $X = \cup_{n\in\mathbb{N}} X_n$ and

$$\int_{X_n} \int_{X_n} \frac{d\mu(x)\ d\mu(y)}{K(x,y)} < +\infty. \tag{18}$$

We start with the following easy lemma :

Proposition 1 *Let f_0 be non-negative and measurable and let f_n be inductively defined as*

$$f_{n+1}(x) = f_0(x) + \int_X K(x,y) f_n^2(y)\ d\mu(y) \tag{19}$$

Let $f = \sup_{n\in\mathbb{N}} f_n(x)$. Then either $f = +\infty$ almost everywhere or $f < +\infty$ almost everywhere. If $f < +\infty$, then f is a solution to equation (17).

Proof : Due to the inequalities $f_0 \geq 0$ and $K \geq 0$, we find by induction that $0 \leq f_n$, so that f_{n+1} is well defined (with values in $[0, +\infty]$); we get moreover (by induction, as well) that $f_n \leq f_{n+1}$. We thus may apply the theorem of monotone convergence and get that $f(x) = f_0(x) + \int_X K(x,y) f^2(y)\ d\mu(y)$. If $f = +\infty$ on a set of positive measure, then $\int_X K(x,y) f^2(y)\ d\mu(y) = +\infty$ almost everywhere and $f = +\infty$ almost everywhere. $\diamond$

We see that if f_0 is such that equation (1) has a solution f which is finite almost everywhere, then we have $f_0 \leq f$ and $\int_X K(x,y) f^2(y)\ d\mu(y) \leq f(x)$. This is almost a characterization of such functions f_0 :

Proposition 2 *Let C_K be the set of non-negative measurable functions Ω such that $\Omega < +\infty$ (almost everywhere) and $\int_X K(x,y)\Omega^2(y)\ d\mu(y) \leq \Omega(x)$. Then, if $\Omega \in C_K$ and if f_0 is a non-negative measurable function such that $f_0 \leq \frac{1}{4}\Omega$, equation (1) has a solution f which is finite almost everywhere.*

Proof : Take the sequence of functions $(f_n)_{n\in\mathbb{N}}$ defined in Proposition 1. By induction, we see that $f_n \leq \frac{1}{2}\Omega$, and thus $f = \sup_n f_n \leq \frac{1}{2}\Omega$. $\diamond$

This remark leads us to define a Banach space of measurable functions in which it is natural to solve equation (1) :

Proposition 3 *Let $\mathcal{E}_K$ be the space of measurable functions f on X such that there exists $\lambda \geq 0$ and $\Omega \in C_K$ such that $|f(x)| \leq \lambda\Omega$ almost everywhere. Then :*

- *$\mathcal{E}_K$ is a linear space*
- *the function $f \in \mathcal{E}_K \mapsto \|f\|_K = \inf\{\lambda \ / \ \exists \Omega \in C_k \ |f| \leq \lambda\Omega\}$ is a semi-norm on $\mathcal{E}_K$*
- *$\|f\|_K = 0 \Leftrightarrow f = 0$ almost everywhere*
- *The normed linear space E_K (obtained from $\mathcal{E}_K$ by quotienting with the relationship $f \sim g \Leftrightarrow f = g$ a.e.) is a Banach space.*
- *If $f_0 \in \mathcal{E}_K$ is non-negative and satisfies $\|f_0\|_K < \frac{1}{4}$, then equation (17) has a non-negative solution $f \in \mathcal{E}_K$.*

Proof : Since $t \mapsto t^2$ is a convex function, we find that C_K is a balanced convex set and thus that $\mathcal{E}_K$ is a linear space and $\| \ \|_K$ is a semi-norm on $\mathcal{E}_K$.

Next, we see that, for $\Omega \in C_K$, $p, q \in \mathbb{N}$, we have

$$\int_{Y_p \cap X_q} \Omega(x) \ d\mu(x) \leq \frac{\int_{X_n} \int_{X_n} \frac{d\mu(x) \ d\mu(y)}{K(x,y)}}{(\mu(Y_p \cap X_q))^2} \tag{20}$$

This is easily checked by writing that

$$\int\int_{(Y_p \cap X_q)^2} \Omega(y) \ d\mu(x) \ d\mu(x) \leq \sqrt{\int_{X_q} \int_{X_q} \frac{d\mu(x) \ d\mu(y)}{K(x,y)}} \sqrt{\int_{Y_p \cap X_q} [\int K(x,y)\Omega^2(y) \ d\mu(y)] \ d\mu(x)} \tag{21}$$

Thus we find that, when $\|f\|_K = 0$, we have $\int \int_{Y_p \cap X_q} |f(x)| \ d\mu(x) = 0$ for all p and q, so that $f = 0$ almost everywhere.

Similarly, we find that if $\lambda_n \geq 0$, $\Omega_n \in C_K$ and $\sum_{n \in \mathbb{N}} \lambda_n = 1$, then, if $\Omega = \sum_{n \in \mathbb{N}} \lambda_n \Omega_n$, we have (by dominated convergence),

$$\int_{Y_p \cap X_q} \Omega(x) \ d\mu(x) \leq \frac{\int_{X_n} \int_{X_n} \frac{d\mu(x) \ d\mu(y)}{K(x,y)}}{(\mu(Y_p \cap X_q))^2} \tag{22}$$

so that $\Omega < +\infty$ almost everywhere. Moreover (by dominated convergence) we have $\Omega \in C_K$. From that, we easily get that E_K is complete.

Finally, existence of a solution of (17) when $\|f_0\|_K < \frac{1}{4}$ is a consequence of Proposition 2. $\diamond$

An easy corollary of Proposition 3 is the following one :

Proposition 4 *If E is a Banach space of measurable functions such that :*

- $f \in E \Rightarrow |f| \in E$ *and* $\| \, |f| \, \|_E \leq C_E \|f\|_E$
- $\| \int_X K(x,y) f^2(y) \, d\mu(y) \|_E \leq C_E \|f\|_E^2$

then E is continuously embedded into E_K.

3 Multipliers.

In this section, we recall a result of Kalton and Verbitsky that characterizes the space E_K for a general class of kernels K.

Theorem 1 (Kalton and Verbitsky [14], Theorem 5.7) *Assume that the kernel K satisfies :*

- $\rho(x,y) = \frac{1}{K(x,y)}$ *is a quasi-metric :*
  1. $\rho(x,y) = \rho(y,x) \geq 0$
  2. $\rho(x,y) = 0 \Leftrightarrow x = y$
  3. $\rho(x,y) \leq \kappa(\rho(x,z) + \rho(z,y))$
- *K satisfies the following inequality : there exists a constant $C > 0$ such that, for all $x \in X$ and all $R > 0$, we have*

$$\int_0^R \int_{\rho(x,y)<t} d\mu(y) \frac{dt}{t^2} \leq CR \int_R^{+\infty} \int_{\rho(x,y)<t} d\mu(y) \frac{dt}{t^3} \tag{23}$$

Then the following assertions are equivalent for a measurable function f on X :

- *(A)* $f \in E_K$
- *(B) There exists a constant C such that, for all $g \in L^2$, we have*

$$\int_X |f(x)|^2 \Big| \int_X K(x,y) g(y) \, d\mu(y) \Big|^2 \, d\mu(x) \leq C \|g\|_2^2 \tag{24}$$

A direct consequence of this theorem is the following one :

Theorem 2 *Let (X, δ, μ) be a space of homogeneous type :*

- *for all $x, y \in X$,* $\delta(x,y) \geq 0$

- *$\delta(x,y) = \delta(y,x)$*
- *$\delta(x,y) = 0 \Leftrightarrow x = y$*
- *there is a positive constant κ such that :*

$$\textit{for all } x, y, z \in X, \delta(x,y) \le \kappa(\delta(x,z) + \delta(z,y)) \tag{25}$$

- *there exists postive A, B and Q which satisfy :*

$$\textit{for all } x \in X, \ \textit{for all } r > 0, Ar^Q \le \int_{\delta(x,y)<r} d\mu(y) \le Br^Q \tag{26}$$

Let

$$K_\alpha(x,y) = \frac{1}{\delta(x,y)^{Q-\alpha}} \tag{27}$$

(where $0 < \alpha < Q/2$) and E_{K_α} the associated Banach space (defined in Proposition 3). Let $\mathcal{I}_\alpha$ be the Riesz operator asociated K_α :

$$\mathcal{I}_\alpha f(x) = \int_X K_\alpha(x,y) f(y) \ d\mu(y). \tag{28}$$

We define two further linear spaces associated to K_α :

- *the Sobolev space W^α defined by*

$$g \in W^\alpha \Leftrightarrow \exists h \in L^2 \ g = \mathcal{I}_\alpha h \tag{29}$$

- *the multiplier space $\mathcal{V}^\alpha$ defined by*

$$f \in \mathcal{V}^\alpha \Leftrightarrow \|f\|_{\mathcal{V}^\alpha} = \Big(\sup_{\|h\|_2 \le 1} \int_X |f(x)|^2 |\mathcal{I}_\alpha h(x)|^2 \ d\mu(x) \Big)^{1/2} < +\infty \tag{30}$$

  (so that pointwise multiplication by a function in $\mathcal{V}^\alpha$ maps boundedly W^α to L^2).

Then we have (with equivalence of norms) for $0 < \alpha < Q/2$:

$$E_{K_\alpha} = \mathcal{V}^\alpha. \tag{31}$$

Proof : It is enough to see that $At^{\frac{Q}{Q-\alpha}} \le \int_{\rho(x,y)<t} d\mu(y) \le Bt^{\frac{Q}{Q-\alpha}}$ (with $\rho(x,y) = \frac{1}{K(x,y)}$) and that $1 < \frac{Q}{Q-\alpha} < 2$, then use Theorem 1. $\diamond$

We shall be interested in two examples :

3.1 Riesz potentials on $\mathbb{R}^n$.

In the case of the usual Euclidean space $\mathbb{R}^n$ with $\delta(x,y) = |x-y| = \sqrt{\sum_{i=1}^n |x_i - y_i|^2}$, we find that W^α is the homogeneous Sobolev space $\dot{H}^\alpha$, i.e. the Banach space of tempered distributions such that their Fourier transforms $\hat{f}$ are locally integrable and satisfy $\int |\xi|^{2\alpha} |\hat{f}(\xi)|^2 \, d\xi < +\infty$.

3.2 Parabolic Riesz potential on $\mathbb{R} \times \mathbb{R}^n$.

We shall use the parabolic (quasi)-distance

$$\delta_\alpha((t,x),(s,y)) = |t-s|^{1/\alpha} + |x-y| \tag{32}$$

on $\mathbb{R} \times \mathbb{R}^n$, where $0 < \alpha$. The associated homogeneous dimension (for the Lebesgue measure) is $Q = n + \alpha$.

For $0 < \beta < \alpha$, we consider the kernel

$$K_{\alpha,\beta}(x,y) = \frac{1}{\delta_\gamma(x,y)^{Q-(\alpha-\beta)}} \tag{33}$$

From Theorem 2, we know that $E_{K_{\alpha,\beta}} = \mathcal{V}^{\alpha,\beta} = \mathcal{M}(W^{\alpha,\beta} \mapsto L^2)$ whenever $0 < \alpha - \beta < Q/2$ (i.e. $\beta < \alpha < n + 2\beta$). In the following section, we shall give a characterization of $W^{\alpha,\beta}$ when $\beta < 2$.

4 The case $\beta < 2$.

We now are going to give a characterization of $W^{\alpha,\beta}(\mathbb{R} \times \mathbb{R}^n)$:

Theorem 3 *If $\beta < 2$, $W^{\alpha,\beta}(\mathbb{R} \times \mathbb{R}^n)$ is the Banach space of tempered distributions such that their Fourier transforms $\hat{f}$ are locally integrable and satisfy*

$$\int\int (|\xi|^{\alpha-\beta} + |\tau|^{1-\frac{\beta}{\alpha}})^2 |\hat{f}(\tau,\xi)|^2 \, d\xi \, d\tau < +\infty \tag{34}$$

Equivalently, we have : $W^{\alpha,\beta}(\mathbb{R} \times \mathbb{R}^n) = L^2_t \dot{H}^{\alpha-\beta}_x \cap L^2_x \dot{H}^{1-\frac{\beta}{\alpha}}_t$.

To prove this theorem, we shall use the theory of γ-stable processes on $\mathbb{R}^p$ for the cases $p = n$ and $\gamma = \beta$, and $p = 1$ and $\gamma = \frac{\alpha-\beta}{\alpha}$.

Let $W_{\gamma,p}(x)$ be defined, for $p \in \mathbb{N}^*$ and $0 < \gamma \leq 2$, as

$$W_{\gamma,p}(x) = \frac{1}{(2\pi)^p} \int_{\mathbb{R}^p} e^{-|\xi|^\gamma} e^{i\,x.\xi} \, d\xi \tag{35}$$

When $\gamma = 2$, we get the Gaussian function

$$W_{2,p}(x) = \frac{1}{(4\pi)^{p/2}} e^{-\frac{|x|^2}{4}}. \tag{36}$$

When $0 < \gamma < 2$, we have a subordination of $W_{\gamma,p}$ to $W_{2,p}$:

$$W_{\gamma,p}(x) = \int_0^{+\infty} \frac{1}{\sigma^{p/2}} W_{2,p}(\frac{x}{\sqrt{\sigma}})\ d\mu_\gamma(\sigma) \tag{37}$$

where $d\mu_\gamma$ is a probability measure on $(0,+\infty)$[25].

We have the following important result of Blumenthal and Getoor [3] :

Lemma 2 *Let $0 < \gamma < 2$. There exists a positive constant $c_{\gamma,p}$ such that*

$$\lim_{|x|\to+\infty} W_{\gamma,p}(x)|x|^{p+\gamma} = c_{\gamma,p}. \tag{38}$$

Thus, we have

$$e^{-t(-\Delta)^{\gamma/2}} f = \int_{\mathbb{R}^p} \frac{1}{t^{\frac{p}{\gamma}}} W_{\gamma,p}(\frac{y}{t^{\frac{1}{\gamma}}}) f(x-y)\ dy \tag{39}$$

with

$$\frac{1}{t^{\frac{p}{\gamma}}} W_{\gamma,p}(\frac{y}{t^{\frac{1}{\gamma}}}) \approx \Omega(\frac{t}{(t^{\frac{1}{\gamma}} + |x|)^{p+\gamma}}) \tag{40}$$

(where $\Omega(.)$ is the Landau notation : $F = \Omega(G)$ if there are two positive constants c_1 and c_2 such that $c_1 < F/G < c_2$).

In order to prove Theorem 3, lt us remark that equation (9) involves a convolution on $\mathbb{R}\times\mathbb{R}^n$ with $K_{\alpha,\beta}$. It will be interesting to give an approximate Fourier transform of the convolution kernel $K_{\alpha,\beta}$.

Proposition 5 *Let $0 < \beta < \min(\alpha, 2)$. Let $\mathbb{K}_{\alpha,\beta}(t,x)$ be defined on $\mathbb{R}\times\mathbb{R}^n$ as*

$$\mathbb{K}_{\alpha,\beta}(t,x) = \frac{1}{|t|^{\frac{n+\beta}{\alpha}}} W_{\beta,n}\left(\frac{x}{|t|^{\frac{1}{\alpha}}}\right) \tag{41}$$

Then :

$$\mathbb{K}_{\alpha,\beta}(t,x) \approx \Omega(K_{\alpha,\beta}(t,x)). \tag{42}$$

Let $M_{\alpha,\beta}(\tau,\xi)$ be the Fourier transform of $\mathbb{K}_{\alpha,\beta}(t,x)$. Then

$$M_{\alpha,\beta}(\tau,\xi) \approx \Omega\left(\frac{1}{|\xi|^{\alpha-\beta} + |\tau|^{1-\frac{\beta}{\alpha}}}\right). \tag{43}$$

Proof : Inequality (42) is a direct consequence of (40) with $\gamma = \beta$ and $p = n$. We then compute the Fourier transform $M_{\alpha,\beta}(\tau,\xi)$ as the Fourier transform in the time variable t of the Fourier transform $N(t,\xi)$ in the space variable x of $\mathbb{K}_{\alpha,\beta}$. We have

$$N(t,\xi) = \frac{1}{|t|^{\frac{\beta}{\alpha}}} e^{-|t|^{\frac{\beta}{\alpha}}|\xi|^\beta} \tag{44}$$

so that

$$M_{\alpha,\beta}(\tau,\xi) = C \int_{\mathbb{R}} \frac{1}{|\tau-\eta|^{1-\frac{\beta}{\alpha}}} \frac{1}{|\xi|^\alpha} W_{\frac{\beta}{\alpha},1}\left(\frac{\eta}{|\xi|^\alpha}\right)\, d\eta \tag{45}$$

Thus, we have

$$M_{\alpha,\beta}(\tau,\xi) \approx \Omega\left(\int_{\mathbb{R}} \frac{1}{|\tau-\eta|^{1-\frac{\beta}{\alpha}}} \frac{|\xi|^\beta}{(|\xi|^\alpha + |\eta|)^{1+\frac{\beta}{\alpha}}}\, d\eta\right). \tag{46}$$

We may rewrite that estimate as

$$M_{\alpha,\beta}(\tau,\xi) \approx \Omega\left(\frac{1}{|\xi|^{\alpha-\beta}} A_{\alpha,\beta}(\frac{\tau}{|\xi|^\alpha})\right) \tag{47}$$

with

$$A_{\alpha,\beta}(\tau) = \int_{\mathbb{R}} \frac{1}{|\tau-\eta|^{1-\frac{\beta}{\alpha}}} \frac{1}{(1+|\eta|)^{1+\frac{\beta}{\alpha}}}\, d\eta. \tag{48}$$

Let $G(\tau) = \frac{1}{|\tau|^{1-\frac{\beta}{\alpha}}}$ and $H(\tau) = \frac{1}{(1+|\tau|)^{1+\frac{\beta}{\alpha}}}$, so that $A_{\alpha,\beta} = G * H$. Since $G \in L^1 + L^\infty(\mathbb{R})$ and $H \in L^1 \cap L^\infty(\mathbb{R})$, we have that $H * G$ is continuous, positive and bounded, so that we have : for $|\tau| \leq 2$, $A_{\alpha,\beta}(\tau) \approx \Omega(1)$. For $|\tau| > 2$, we write :

- $H * G(\tau) \geq \left(\frac{2}{|\tau|}\right)^{1-\frac{\beta}{\alpha}} \int_{-1}^{1} H(\eta)\, d\eta$
- $\int_{-|\tau|/2}^{|\tau|/2} G(\tau-\eta)H(\eta)\, d\eta \leq \left(\frac{2}{|\tau|}\right)^{1-\frac{\beta}{\alpha}} \|H\|_1$
- $\int_{|\eta|>|\tau|/2} G(\tau-\eta)H(\eta)\, d\eta \leq \int_{|\eta|>|\tau|/2} \frac{1}{|\tau-\eta|^{1-\frac{\beta}{\alpha}}} \frac{1}{|\eta|^{1+\frac{\beta}{\alpha}}}\, d\eta = C\frac{1}{|\tau|} \leq C\left(\frac{1}{|\tau|}\right)^{1-\frac{\beta}{\alpha}}$

so that $A_{\alpha,\beta}(\tau) \approx \Omega\left(\frac{1}{|\tau|^{1-\frac{\beta}{\alpha}}}\right)$. ◇

Now, Theorem 3 is a direct consequence of Proposition 5.

5 Parabolic Morrey spaces and the Fefferman–Phong inequality.

We follow in this section the notations of Theorem 2 : (X, δ, μ) is a space of homogeneous type, with homogeneous dimension Q. For $0 < \alpha < Q/2$, $\mathcal{I}_\alpha$ is the Riesz potential associated to the kernel $K_\alpha = \frac{1}{\delta(x,y)^{Q-\alpha}}$, and $\mathcal{V}^\alpha$ is the space of functions that satisfy

$$\|f\|_{\mathcal{V}^\alpha} = \Big(\sup_{\|h\|_2 \leq 1} \int_X |f(x)|^2 |\mathcal{I}_\alpha h(x)|^2 \, d\mu(x)\Big)^{1/2} < +\infty. \tag{49}$$

We saw that $\mathcal{V}^\alpha$ is the space of pointwise multipliers who map $W^\alpha = \mathcal{I}_\alpha L^2$ to L^2. This space of multipliers is not easy to handle (it can be characterized through capacitary inequalities, see [23] for the Euclidean case). So, we will use some spaces that are very close to $\mathcal{V}^\alpha$: the (homogeneous) Morrey–Campanato spaces.

Definition 1 *The (homogeneous) Morrey–Campanato space* $\dot{M}^{p,q}(X)$ $(1 < p \leq q < +\infty)$ *is the space of the functions that are locally* L^p *and satisfy*

$$\|f\|_{\dot{M}^{p,q}} = \sup_{x \in X} \sup_{R > 0} R^{Q(\frac{1}{q} - \frac{1}{p})} \Big(\int_{\delta(x,y) < R} |f(y)|^p \, d\mu(y)\Big)^{1/p} < +\infty. \tag{50}$$

Remark that $L^q \subset \dot{M}^{p,q}(X)$, as it is easy to check by using Hölder inequality.

We shall need two technical lemmas on Morrey–Campanato spaces. The first lemma deals with the Hardy–Littlewood maximal function :

Lemma 3 *Let* $\mathcal{M}_f$ *be the Hardy–Littlewood maximal function of* f *:*

$$\mathcal{M}_f(x) = \sup_{R > 0} \frac{1}{\mu(B(x,R))} \int_{B(x,R)} |f(y)| \, d\mu(y) \tag{51}$$

where $B(x,R) = \{y \in X \ / \ \delta(x,y) < R\}$. *Then there exists constants* C_p *and* $C_{p,q}$ *such that :*

- *for every* $f \in L^1$ *and every* $\lambda > 0$,

$$\mu(\{x \in X \ / \ \mathcal{M}_f(x) > \lambda\}) \leq C_1 \frac{\|f\|_1}{\lambda}$$

- *for* $1 < p \leq +\infty$ *and for every* $f \in L^p$

$$\|\mathcal{M}_f\|_p \leq C_p \|f\|_p$$

- *for every* $1 < p \leq q < +\infty$ *and for every* $f \in \dot{M}^{p,q}(X)$

$$\|\mathcal{M}_f\|_{\dot{M}^{p,q}} \leq C_{p,q}\|f\|_{\dot{M}^{p,q}}$$

Proof : The weak type (1,1) of the Hardy–Littlewood maximal function is a classical result (see Coifman and Weiss [7] for the spaces of homogeneous type). The boundedness of the maximal function on L^p for $1 < p \leq +\infty$ is then a direct consequence of the Marcinkiewicz interpolation theorem [11].

Thus, we shall be interested in the proof for $\dot{M}^{p,q}(X)$. Let $f \in \dot{M}^{p,q}(X)$. For $x \in X$ and $R > 0$, we need to estimate $\int_{B(x,R)} |\mathcal{M}_f(y)|^p \, d\mu(y)$. We write $f = f_1 + f_2$, where $f_1(y) = f(y)1_{B(x,2\kappa R)}(y)$. We have $\mathcal{M}_f \leq \mathcal{M}_{f_1} + \mathcal{M}_{f_2}$. We have

$$\int_{B(x,R)} \mathcal{M}_{f_1}(y)^p \, d\mu(y) \leq (C_p\|f_1\|_p)^p \leq C_p^p\|f\|^p_{\dot{M}^{p,q}}(2\kappa R)^{Q(1-\frac{p}{q})}.$$

On the other hand, for $\delta(x,y) \leq R$,

$$\mathcal{M}_{f_2}(y) = \sup_{\rho > R} \frac{1}{\mu(B(y,\rho))}\int_{B(y,\rho)} |f_2(z)| \, d\mu(z) \leq \sup_{\rho > R} \frac{1}{A\rho^Q}\|f\|_{\dot{M}^{p,q}}\rho^{Q(1-\frac{1}{q})}$$

so that $1_{B(x,R)}\mathcal{M}_{f_2} \leq \frac{\|f\|_{\dot{M}^{p,q}}}{AR^{\frac{1}{q}}}$ and

$$\int_{B(x,R)} \mathcal{M}_{f_2}(y)^p \, d\mu(y) \leq \mu(B(x,R))\|1_{B(x,R)}\mathcal{M}_{f_2}\|^p_\infty \leq \frac{B}{A^p}\|f\|^p_{\dot{M}^{p,q}}R^{Q(1-\frac{p}{q})}.$$

◇

The second lemma is a pointwise estimate for the Riesz potential, known as the *Hedberg inequality* [12, 1].

Lemma 4 *If* $f \in \dot{M}^{p,q}(X)$ *and if* $0 < \alpha < \frac{Q}{q}$, *then*

$$|\int_X \frac{1}{\delta(x,y)^{Q-\alpha}} f(y) \, d\mu(y)| \leq C_{p,q,\alpha}(\mathcal{M}_f(x))^{1-\frac{\alpha q}{Q}}\|f\|^{\frac{\alpha q}{Q}}_{\dot{M}^{p,q}}. \tag{52}$$

Proof : Let $R > 0$. We have

$$\begin{aligned} |\int_{\rho(x,y)<R} \frac{f(y)}{\delta(x,y)^{Q-\alpha}} d\mu(y)| &\leq \sum_{j=0}^{+\infty} \int_{\frac{R}{2^{j+1}} \leq \rho(x,y) < \frac{R}{2^j}} \frac{|f(y)|}{\delta(x,y)^{Q-\alpha}} \, d\mu(y) \\ &\leq \sum_{j=0}^{+\infty} B2^{-j\alpha}R^\alpha \frac{1}{\mu(B(x,2^{-j}R))}\int_{B(x,2^{-j}R)} |f(y)| \, d\mu(y) \\ &\leq B\frac{1}{1-2^{-\alpha}}R^\alpha \mathcal{M}_f(x) \end{aligned}$$

and

$$|\int_{\rho(x,y)\geq R}\frac{f(y)}{\delta(x,y)^{Q-\alpha}}d\mu(y)|\leq\sum_{j=0}^{+\infty}\int_{2^jR\leq\rho(x,y)<2^{j+1}R}\frac{|f(y)|}{\delta(x,y)^{Q-\alpha}}\ d\mu(y)$$

$$\leq\sum_{j=0}^{+\infty}\frac{1}{(2^jR)^{Q-\alpha}}B^{1-\frac{1}{p}}(2^{j+1}R)^{Q(1-\frac{1}{p})}(2^{j+1}R)^{Q(\frac{1}{p}-\frac{1}{q})}\|f\|_{\dot M^{p,q}}$$

$$\leq B^{1-\frac{1}{p}}\frac{2^{Q(1-\frac{1}{q})}}{1-2^{\alpha-\frac{Q}{q}}}R^{\alpha-\frac{Q}{q}}\|f\|_{\dot M^{p,q}}$$

We then end the proof by taking $R^{\frac{Q}{q}}=\frac{\|f\|_{\dot M^{p,q}}}{\mathcal{M}_f(x)}$. ◇

As a direct corollary of Lemma 4, we get the following result of Adams [2] on Riesz potentials :

Corollary 1 *For $0<\alpha<\frac{Q}{q}$, the Riesz potential $\mathcal{I}_\alpha$ is bounded from $\dot M^{p,q}(X)$ to $\dot M^{\frac{p}{\lambda},\frac{q}{\lambda}}(X)$, with $\lambda=1-\frac{\alpha q}{Q}$.*

We may now state the comparison result between spaces of multipliers and Morrey–Campanato spaces, a result which is known as the *Fefferman–Phong inequality* [9] :

Theorem 4 *Let $0<\alpha<Q/2$ and $2<p\leq\frac{Q}{\alpha}$. Then we have :*

$$\dot M^{p,\frac{Q}{\alpha}}(X)\subset\mathcal{V}^\alpha=\mathcal{M}(W^\alpha\mapsto L^2)\subset\dot M^{2,\frac{Q}{\alpha}}(X)\tag{53}$$

Proof : For $f\in\dot M^{p,\frac{Q}{\alpha}}(X)$ and $g\in\dot M^{p,\frac{Q}{\alpha}}(X)$, we have $fg\in\dot M^{\frac{p}{2},\frac{Q}{2\alpha}}(X)$. We have $p/2>1$ and $\alpha<Q/q$ with $q=\frac{Q}{2\alpha}$, hence, since $\lambda=1-\frac{\alpha q}{Q}=1/2$, $\mathcal{I}_\alpha(fg)\in\dot M^{p,q}(X)$. Thus, from Proposition 4, we see that $\dot M^{p,\frac{Q}{\alpha}}(X)\subset\mathcal{V}^\alpha$.

The embedding $\mathcal{V}^\alpha\subset\dot M^{2,\frac{Q}{\alpha}}(X)$ is easy to check. Indeed, if $F=1_{B(x,2\kappa R)}$, we have for $y\in B(x,R)$

$$\mathcal{I}_\alpha F(y)\geq\int_{\rho(z,y)<R}\frac{d\mu(z)}{\rho(z,y)^{Q-\alpha}}\geq\frac{\mu(B(y,R)}{R^{Q-\alpha}}\geq AR^\alpha$$

hence, for $f\in\mathcal{V}^\alpha$,

$$\int_{B(x,R)}|f(y)|^2\ d\mu(y)\leq\frac{1}{A^2R^{2\alpha}}\|f\|^2_{\mathcal{V}^\alpha}\|F\|_2^2\leq\frac{B}{A^2}\|f\|^2_{\mathcal{V}^\alpha}R^{Q-2\alpha}.$$

◇

Remark : The embeddings are strict. For a proof in the case of the Euclidean space, see for instance [19].

In the following, we shall write $\dot M^{p,q}(\mathbb{R}^n)$ when we have $X=\mathbb{R}^n$ with $\delta(x,y)=|x-y|$, and $\dot M^{p,q}_\alpha(\mathbb{R}\times\mathbb{R}^n)$ when we have $X=\mathbb{R}^n$ with $\delta_\alpha((t,x),(s,y))=|t-s|^{1/\alpha}+|x-y|$.

6 Cheap solutions for a semilinear parabolic equation.

In this section we consider our Cauchy problem : given $\vec{u}_0 \in (\mathcal{S}'(\mathbb{R}^n)^d$, find a vector distribution $\vec{u}$ on $(0,+\infty)\times\mathbb{R}^n$ so that, for $i=1,\ldots,d$ we have

$$\partial_t u_i = -(-\Delta)^{\alpha/2} u_i + \sum_{j=1}^d \sum_{k=1}^d \sigma_{i,j,k}(D)(u_j u_k) \tag{54}$$

and

$$\lim_{t\to 0} u_i(t,x) = u_{i,0}. \tag{55}$$

where $\sigma_{i,j,k}(D)$ is an homogeneous pseudo-differential operator of dergee β with $0<\beta<\alpha<n+2\beta$.

Due to Lemma 1, we have a domination principle :

Theorem 5 *If there exist a function $W(t,s)$ such that*

$$\iint_{\mathbb{R}\times\mathbb{R}^n} \frac{1}{\left(|t-s|^{\frac{1}{\alpha}}+|x-y|\right)^{n+\beta}} W^2(s,y)\ ds\ dy \le W(t,x) \tag{56}$$

and such that, for some $T\in(0,+\infty]$ we have

$$1_{0<t<T}|e^{-t(-\Delta)^{\alpha/2}}\vec{u}_0| \le \frac{1}{4C_0} W \tag{57}$$

[where C_0 is the constant given in Lemma 1], then defining inductively $\vec{U}_k$ on $(0,T)\times\mathbb{R}^n$ and W_k on $\mathbb{R}\times\mathbb{R}^n$ as

- $\vec{U}_0(t,x) = e^{-t(-\Delta)^{\alpha/2}}\vec{u}_0$ *for* $0<t<T$
- $W_0(t,x) = 1_{0<t<T}|e^{-t(-\Delta)^{\alpha/2}}\vec{u}_0|$
- $\vec{U}_{k+1} = \vec{U}_0 + \int_0^t e^{-(t-s)(-\Delta)^{\alpha/2}}\sigma(D)(\vec{U}_k\otimes\vec{U}_k)\ ds$
- $W_{k+1} = W_0 + \iint_{\mathbb{R}\times\mathbb{R}^n} \frac{C_0}{\left(|t-s|^{\frac{1}{\alpha}}+|x-y|\right)^{n+\beta}} W_k(s,y)^2\ ds\ dy$

we have the following results :

- W_k *converges monotonically to a function* W_∞ *such that* $W_\infty \le \frac{1}{2C_0}W$ *and*

$$W_\infty = W_0 + \iint_{\mathbb{R}\times\mathbb{R}^n} \frac{C_0}{\left(|t-s|^{\frac{1}{\alpha}}+|x-y|\right)^{n+\beta}} W_\infty(s,y)^2\ ds\ dy \tag{58}$$

- $|\vec{U}_0| \leq W_0$ *and* $|\vec{U}_{k+1} - \vec{U}_k| \leq W_{k+1} - W_k$ *on* $(0,T) \times \mathbb{R}^n$
- *the sequence* $(\vec{U}_k(t,x))_{k\in\mathbb{N}}$ *converges pointwise to a solution* $\vec{U}_\infty$ *of*

$$\vec{U}_\infty = \vec{U}_0 + \int_0^t e^{-(t-s)(-\Delta)^{\alpha/2}} \sigma(D)(\vec{U}_\infty \otimes \vec{U}_\infty)\, ds \tag{59}$$

As we did not use any refined analysis of the coefficients $\sigma_{i,j,k}(D)$ (no maximum principle, no conservation of energy, and so on), but just controlled the integrals by the absolute values of the integrands, we shall call the solutions we found as *cheap solutions* : they do not provide much insight on the structure of the equation.

Instead of considering pointwise estimates, we can give the same proof in the setting of the Banach contraction principle and find the following results on global or local existence of solutions :

Theorem 6 *Let* $\mathcal{V}^{\alpha,\beta}(\mathbb{R} \times \mathbb{R}^n)$ *be the space* $\mathcal{M}(W^{\alpha,\beta} \mapsto L^2)$ *described in Subsection 3.2 and* $\| \ \|_{K_{\alpha,\beta}}$ *be the norm on* $\mathcal{V}^{\alpha,\beta}$ *described in Proposition 3. Then if* $\vec{u}_0$ *is such that*

$$1_{t>0}|e^{-t(-\Delta)^{\alpha/2}}\vec{u}_0| \in \mathcal{V}^{\alpha,\beta} \text{ and } \|1_{t>0}|e^{-t(-\Delta)^{\alpha/2}}\vec{u}_0|\|_{K_{\alpha,\beta}} < \frac{1}{4C_0} \tag{60}$$

(where C_0 *is the constant in Lemma 1), then the equation*

$$\vec{u} = e^{-t(-\Delta)^{\alpha/2}}\vec{u}_0 + \int_0^t e^{-(t-s)(-\Delta)^{\alpha/2}} \sigma(D)(\vec{u} \otimes \vec{u})\, ds. \tag{61}$$

has a solution $\vec{u}$ *on* $(0,+\infty) \times \mathbb{R}^n$ *such that* $1_{t>0}\vec{u} \in (\mathcal{V}^{\alpha,\beta})^d$.

Proof : We define an operator $\mathcal{B}$ on $(\mathcal{V}^{\alpha,\beta})^d$ by

$$B(\vec{u},\vec{v}) = \int_{-\infty}^t e^{-(t-s)(-\Delta)^{\alpha/2}} \sigma(D)(\vec{u} \otimes \vec{v})\, ds. \tag{62}$$

and we are going to solve $\vec{U} = \vec{U}_0 + \mathcal{B}(\vec{U},\vec{U})$ with $\vec{U}_0 = 1_{t>0}e^{-t(-\Delta)^{\alpha/2}}\vec{u}_0$.

We have, from Lemma 1, that

$$|\mathcal{B}(\vec{U},\vec{V})| \leq C_0 \int_{\mathbb{R}\times\mathbb{R}^n} K_{\alpha,\beta}(t-s,x-y)|\vec{U}(s,y)|\,|\vec{V}(s,y)|\, ds\, dy \tag{63}$$

so that

$$\|\mathcal{B}(\vec{U},\vec{V})\|_{K_{\alpha,\beta}} \leq C_0 \|\vec{U}\|_{K_{\alpha,\beta}} \|\vec{V}\|_{K_{\alpha,\beta}} \tag{64}$$

The Banach contraction principle gives that, when $\|\vec{U}_0\|_{K_{\alpha,\beta}} < \frac{1}{4C_0}$, there exists a unique solution $\vec{U}$ such that $\|\vec{U}\|_{K_{\alpha,\beta}} < \frac{1}{2C_0}$. For $\vec{u}_0$ satisfying the assumptions of Theorem 6, we thus can find a solution $\vec{U}$ of $\vec{U} = \vec{U}_0 + \mathcal{B}(\vec{U}, \vec{U})$ with $\vec{U}_0 = 1_{t>0}\, e^{-t(-\Delta)^{\alpha/2}} \vec{u}_0$; this solution, obtained by iteration, satisfies $\vec{U} = 0$ for $t < 0$. The solution $\vec{u}$ of Theorem 6 is then given by $\vec{u} = 1_{t>0}\vec{U}$. $\diamond$

Theorem 7 *Let $\mathcal{V}^{\alpha,\beta}(\mathbb{R}\times\mathbb{R}^n)$ be the space $\mathcal{M}(W^{\alpha,\beta} \mapsto L^2)$ described in Subsection 3.2 and $\| \ \|_{K_{\alpha,\beta}}$ be the norm on $\mathcal{V}^{\alpha,\beta}$ described in Proposition 3. Then if $\vec{u}_0$ is such that $1_{0<t<T}|e^{-t(-\Delta)^{\alpha/2}}\vec{u}_0| \in \mathcal{V}^{\alpha,\beta}$ and $\|1_{0<t<T}|e^{-t(-\Delta)^{\alpha/2}}\vec{u}_0|\|_{K_{\alpha,\beta}} < \frac{1}{4C_0}$, then the equation*

$$\vec{u} = e^{-t(-\Delta)^{\alpha/2}}\vec{u}_0 + \int_0^t e^{-(t-s)(-\Delta)^{\alpha/2}}\sigma(D)(\vec{u}\otimes\vec{u})\ ds \tag{65}$$

has a solution $\vec{u}$ on $(0,T)\times\mathbb{R}^n$ such that $1_{0<t<T}\vec{u} \in (\mathcal{V}^{\alpha,\beta})^d$.

Proof : Just solve $\vec{U} = \vec{U}_0 + \mathcal{B}(\vec{U}, \vec{U})$ with $\vec{U}_0 = 1_{0<t<T}\, e^{-t(-\Delta)^{\alpha/2}}\vec{u}_0$. $\diamond$

7 Regularity of the solutions.

In this section, we discuss the size and regularity of global cheap solutions.

Definition 2 *A tempered distribution f wil be said to be low-frequentially bounded if, for every $\varphi \in \mathcal{D}(\mathbb{R}^n)$, we have $\mathcal{F}^{-1}(\varphi\hat{f}) \in L^\infty$, where $\hat{f} = \mathcal{F}f$ is the Fourier transform of f.*

The space $X_{\alpha,\beta}(\mathbb{R}^n)$ is defined as the space of low-frequentially bounded tempered distributions f such that $1_{t>0}\, e^{-t(-\Delta)^{\alpha/2}} f \in \mathcal{V}^{\alpha,\beta}$. It is normed by $\|f\|_{X_{\alpha,\beta}(\mathbb{R}^n)} = \|1_{t>0}\, e^{-t(-\Delta)^{\alpha/2}} f\|_{\mathcal{V}^{\alpha,\beta}}$.

Remark : If f is low-frequentially bounded, then the distribution $e^{-t(-\Delta)^{\alpha/2}}f$ is well-defined for fixed $t > 0$, moreover the function $(t,x) \mapsto e^{-t(-\Delta)^{\alpha/2}}f(x)$ is $\mathcal{C}^\infty$ on $(0,+\infty)\times\mathbb{R}^n$.

We have the following easy result on $X_{\alpha,\beta}(\mathbb{R}^n)$:

Proposition 6 *If $f \in X_{\alpha,\beta}(\mathbb{R}^n)$, $x_0 \in \mathbb{R}^n$ and $\lambda > 0$, then the function f_{λ,x_0} defined by $\lambda^{\alpha-\beta}f(\lambda(x-x_0)) = f_{\lambda,x_0}(x)$ belongs to $X_{\alpha,\beta}(\mathbb{R}^n)$ and we have $\|f_{\lambda,x_0}\|_{X_{\alpha,\beta}(\mathbb{R}^n)} = \|f\|_{X_{\alpha,\beta}(\mathbb{R}^n)}$.*

Corollary 2 *$X_{\alpha,\beta}(\mathbb{R}^n) \subset \dot{B}^{\beta-\alpha}_{\infty,\infty}$ and we have, for a constant C which depends only on n, α and β :*

$$\|e^{-t(-\Delta)^{\alpha/2}} f\|_{L^\infty(dx)} \leq C\|f\|_{X_{\alpha,\beta}(\mathbb{R}^n)} t^{-1+\frac{\beta}{\alpha}} \tag{66}$$

In order to use this last estimate, we shall now prove the following lemma :

Lemma 5 *There exists a constant C which depends only on n, α and β such that :*

$$\begin{aligned} &|t|^{1-\frac{\beta}{\alpha}} | \iint K_{\alpha,\beta}(t-s, x-y) W^2(s,y)\, ds\, dy| \\ &\leq C\|W\|_{\mathcal{V}^{\alpha,\beta}} (\|W\|_{\mathcal{V}^{\alpha,\beta}} + \sup_{s\in\mathbb{R}} |s|^{1-\frac{\beta}{\alpha}} \|W(s,.)\|_\infty) \end{aligned} \tag{67}$$

Proof : The proof is based on the following remark : the function

$$J(t,x) = \iint_{|s|>|t|} \frac{1}{(|t-s|^{\frac{1}{\alpha}} + |x-y|)^{n+\beta}} \frac{1}{(|s|^{\frac{1}{\alpha}} + |y|)^{\alpha+\frac{n-\beta}{2}}}\, ds\, dy \tag{68}$$

is well-defined for $(t,x) \neq (0,0)$, as $\beta < \alpha$ (local integrability) and $\frac{n+\beta}{2} > 0$ (integrability at infinity). By Fatou's lemma, it is semi-continuous, hence, since $\{(t,x) \;/\; \rho_\alpha(t,x) = 1\}$ is a compact set, we have

$$\gamma = \inf_{\rho_\alpha(t,x)=1} J(t,x) > 0 \tag{69}$$

By homogeneity, we find

$$J(t,x) \geq \gamma \frac{1}{\rho_\alpha(t,x)^{(n+\beta)/2}} \tag{70}$$

We may now estimate $I(t,x) = \iint K_{\alpha,\beta}(t-s, x-y) W^2(s,y)\, ds\, dy$. Let $\epsilon \in (0, 1/2)$ and let

$$A_\epsilon(t,x) = \iint_{|t-s|<\epsilon|t|} K_{\alpha,\beta}(t-s, x-y) W^2(s,y)\, ds\, dy \tag{71}$$

and $B_\epsilon(t,x) = I(t,x) - A_\epsilon(t,x)$. Let us define moreover $N_1 = \|W\|_{\mathcal{V}^{\alpha,\beta}}$ and $N_2 = \sup_{s\in\mathbb{R}} |s|^{1-\frac{\beta}{\alpha}} \|W(s,.)\|_\infty$. We have

$$A_\epsilon(t,x) \leq N_2^2 \left(\frac{2}{|t|}\right)^{2-\frac{2\beta}{\alpha}} \iint_{|t-s|<\epsilon t} K_{\alpha,\beta}(t-s, x-y)\, ds\, dy = C N_2^2 \left(\frac{\epsilon}{|t|}\right)^{1-\frac{\beta}{\alpha}} \tag{72}$$

On the other hand, writing $J_\epsilon(t,x) = 1_{|t-s|>\epsilon|t|} J(t-s,x-y)$, we have

$$B_\epsilon(t,x) \le \frac{1}{\gamma^2} \iint J_\epsilon(t-s,x-y)^2 W^2(s,y)\,ds\,dy \tag{73}$$

and

$$J_\epsilon(t-s,x-y) \le \iint \frac{1}{(|s-\sigma|^{\frac{1}{\alpha}}+|y-z|)^{n+\beta}} \frac{1_{|t-\sigma|>\epsilon|t|}}{(|t-\sigma|^{\frac{1}{\alpha}}+|z-x|)^{\alpha+\frac{n-\beta}{2}}}\,d\sigma\,dz \tag{74}$$

Let $F_{t,x,\epsilon}(\sigma,z) = \frac{1_{|t-\sigma|>\epsilon|t|}}{(|t-\sigma|^{\frac{1}{\alpha}}+|z-x|)^{\alpha+\frac{n-\beta}{2}}}$; we get

$$B_\epsilon(t,x) \le \frac{1}{\gamma^2} N_1^2 \iint |F_{t,x,\epsilon}(\sigma,z)|^2\,d\sigma\,dz = C N_1^2 \frac{1}{(\epsilon|t|)^{1-\frac{\beta}{\alpha}}}. \tag{75}$$

We conclude the proof by taking $\epsilon^{1-\frac{\beta}{\alpha}} = \frac{1}{2}\frac{N_1}{N_1+N_2}$. ◇

We now consider a solution $\vec{u}$ on $(0,+\infty)\times\mathbb{R}^n$ of the semi-linear heat equation

$$\vec{u} = e^{-t(-\Delta)^{\alpha/2}}\vec{u}_0 + \int_0^t e^{-(t-s)(-\Delta)^{\alpha/2}}\sigma(D)(\vec{u}\otimes\vec{u})\,ds \tag{76}$$

obtained by the iteration algorithm :

$$\vec{U}_0 = e^{-t(-\Delta)^{\alpha/2}}\vec{u}_0 \text{ and } \vec{U}_{k+1} = \vec{U}_0 + \int_0^t e^{-(t-s)(-\Delta)^{\alpha/2}}\sigma(D)(\vec{U}_k\otimes\vec{U}_k)\,ds. \tag{77}$$

We already know that, if $1_{0<t}\vec{U}_0$ is small enough in $(\mathcal{V}^{\alpha,\beta}(\mathbb{R}\times\mathbb{R}^n)^d$ (i.e. if $\vec{u}_0$ is small enough in $(X_{\alpha,\beta}(\mathbb{R}^n))^d$), then $\sum_{k=0}^{+\infty}\|1_{0<t}(\vec{U}_{k+1}-\vec{U}_k)\|_{\mathcal{V}^{\alpha,\beta}} < +\infty$. We get other estimates from this inequality :

Proposition 7 *If*

$$\|1_{0<t}\,\vec{U}_0\|_{\mathcal{V}^{\alpha,\beta}} + \sum_{k=0}^{+\infty}\|1_{0<t}(\vec{U}_{k+1}-\vec{U}_k)\|_{\mathcal{V}^{\alpha,\beta}} < +\infty, \tag{78}$$

then

$$\sup_{0<t} t^{1-\frac{\beta}{\alpha}}\|\vec{U}_0(t,.)\|_\infty + \sum_{k=0}^{+\infty}\sup_{0<t} t^{1-\frac{\beta}{\alpha}}\|\vec{U}_{k+1}(t,.)-\vec{U}_k(t,.)\|_\infty < +\infty. \tag{79}$$

Proof : Writing $\vec{U}_{-1}=0$, $A_k=|\vec{U}_k-\vec{U}_{k-1}|$ and $B_k=|\vec{U}_k|$, we have, for all $k\in\mathbb{N}$,

$$A_{k+1}(t,x)\leq C_0\int_0^t\int\frac{A_k(s,y)(B_k(s,y)+B_{k-1}(s,y))}{(|t-s|^{1/\alpha}+|x-y|)^{n+\beta}}\,ds\,dy \tag{80}$$

Let us define

$$\begin{aligned}\alpha_k&=\sup_{0<t}t^{1-\frac{\beta}{\alpha}}\|A_k(t,.)\|_\infty\\ \beta_k&=\sup_{0<t}t^{1-\frac{\beta}{\alpha}}\|B_k(t,.)\|_\infty\\ \gamma_k&=\|A_k\|_{\mathcal{V}^{\alpha,\beta}}\\ \delta&=\|B_k\|_{\mathcal{V}^{\alpha,\beta}}.\end{aligned} \tag{81}$$

We remark that $\|\sqrt{|FG|}\|_\infty\leq\sqrt{\|F\|_\infty\|G\|_\infty}$ and $\|\sqrt{|FG|}\|_{\mathcal{V}^{\alpha,\beta}}\leq\sqrt{\|F\|_{\mathcal{V}^{\alpha,\beta}}\|G\|_{\mathcal{V}^{\alpha,\beta}}}$, thus we may apply Lemma 5 and get :

$$\alpha_{k+1}\leq C\sqrt{\alpha_k(\beta_k+\beta_{k-1})\gamma_k(\delta_k+\delta_{k-1})} \tag{82}$$

Let

$$\epsilon_k=\sum_{j\leq k}\alpha_j \text{ and } M=\sum_{k\in\mathbb{N}}\gamma_k \tag{83}$$

We have the inequality

$$\alpha_{k+1}\leq\frac{1}{2}\alpha_k+CM\gamma_k\epsilon_k \tag{84}$$

which gives

$$\epsilon_{k+1}\leq 2CM\sum_{j\leq k}\gamma_j\epsilon_j \tag{85}$$

hence

$$\sum_{j\leq k+1}\gamma_j\epsilon_j\leq(1+2CM\gamma_{k+1})\sum_{j\leq k}\gamma_j\epsilon_j \tag{86}$$

which gives

$$\sum_{j\leq k+1}\gamma_j\epsilon_j\leq\gamma_0\epsilon_0\prod_{l=1}^{+\infty}(1+2CM\gamma_l) \tag{87}$$

and finally

$$\sup_{k\in\mathbb{N}}\epsilon_k\leq 2CM\gamma_0\epsilon_0\prod_{l=1}^{+\infty}(1+2CM\gamma_l) \tag{88}$$

Proposition 7 is proved. ◇

Proposition 8 *Under the same assumptions as in Proposition 7, we have, for all positive γ, that*

$$\sup_{0<t} t^{\frac{\alpha-\beta+\gamma}{\alpha}} \|\vec{u}(t,.)\|_{\dot{B}^{\gamma}_{\infty,\infty}} < +\infty. \tag{89}$$

Hence the solution $\vec{u}$ is $\mathcal{C}^\infty$ on $(0,T)\times\mathbb{R}^n$.

Proof : Let $\gamma \geq 0$. Start from the information that $\sup_{0<t} t^{\frac{\alpha-\beta+\gamma}{\alpha}} \|\vec{u}(t,.)\|_{\dot{B}^{\gamma}_{\infty,\infty}}$ if $\gamma > 0$ and that $\sup_{0<t} t^{\frac{\alpha-\beta}{\alpha}} \|\vec{u}(t,.)\|_\infty < +\infty$. We then have the estimate $\sup_{0<t} t^{\frac{\alpha-\beta+\gamma}{\alpha}} \|\vec{u}(t,.)\otimes\vec{u}(t,.)\|_{\dot{B}^{\gamma}_{\infty,\infty}} < +\infty$. Then write

$$\vec{u}(t,x) = e^{-\frac{t}{2}(-\Delta)^{\alpha/2}}\vec{u}(\frac{t}{2},x) + \int_{t/2}^{t} e^{-(t-s)(-\Delta)^{\alpha/2}}\sigma(D)(\vec{u}(s,.)\otimes\vec{u}(s,.))\,ds \tag{90}$$

to control the the norm of $\vec{u}$ in $\dot{B}^{\gamma+\alpha-\beta}_{\infty,\infty}$. ◇

8 A Besov-space approach of cheap solutions.

Theorems 6 and 7 give a criterion to grant existence of a solution : the initial value is required to satisfy $1_{0<t<T}|e^{-t(-\Delta)^{\alpha/2}}\vec{u}_0| \in \mathcal{V}^{\alpha,\beta}$. But the space of the distributions such that $1_{0<t<T}|e^{-t(-\Delta)^{\alpha/2}}\vec{u}_0| \in \mathcal{V}^{\alpha,\beta}$ is not a classical one and we might try and find some subspaces that are close enough to this maximal space but belong to a classical scale of spaces.

Thus, we shall describe Banach spaces X of measurable functions in time and space variables that lead to cheap solutions : one should have the following properties :

- if $f(t,x) \in X$ and if $|g(t,x)| \leq |f(t,x)|$, then $g \in X$ and $\|g\|_X \leq \|f\|_X$
- for $f,g \in X$, $F = \iint K_{\alpha,\beta}(t-s,x-y)|f(s,y)|\,|g(s,y)|\,ds\,dy \in X$ and $\|F\|_X \leq C_X \|f\|_X \|g\|_X$

From Proposition 4, we know that $X \subset \mathcal{V}^{\alpha,\beta}(\mathbb{R}\times\mathbb{R}^n)$ and from Lemma 1 we know that we may find a solution $\vec{u}$ of the equation

$$\vec{u} = e^{-t(-\Delta)^{\alpha/2}}\vec{u}_0 + \int_0^t e^{-(t-s)(-\Delta)^{\alpha/2}}\sigma(D)(\vec{u}\otimes\vec{u})\,ds \tag{91}$$

on $(0,T)\times\mathbb{R}^n$ such that $1_{0<t<T}\vec{u} \in X^d$ as soon as $1_{0<t<T}|e^{-t(-\Delta)^{\alpha/2}}\vec{u}_0| \in X$ and $\|1_{t0<t<T}|e^{-t(-\Delta)^{\alpha/2}}\vec{u}_0|\|_X < \frac{1}{4C_0C_X}$ (where T might be a positive real number [local solution] or equal to $+\infty$ [global solution]).

The simplest way to find such a space X is to replace the kernel $K_{\alpha,\beta}$ by kernels whose action are well documented on functions in time variable or in space variable. For instance, if $\max(1/2, \beta/\alpha) < \gamma < \min(1, \frac{n+2\beta}{2\alpha})$, we may write

$$K_{\alpha,\beta}(t,x) \leq \frac{1}{|t-s|^\gamma} \frac{1}{|x|^{n+\beta-\alpha\gamma}} \tag{92}$$

Let $I_{x,\alpha\gamma-\beta}$ be the convolution operator (in x variable) with $\frac{1}{|x|^{n+\beta-\alpha\gamma}}$ and $I_{t,1-\gamma}$ be the convolution operator (in t variable) with $\frac{1}{|t|^\gamma}$. We have :

$$\iint K_{\alpha,\beta}(t-s, x-y)|f(s,y)|\,|g(s,y)|\; ds\; dy \leq I_{t,1-\gamma} I_{x,\alpha\gamma-\beta}(|fg|)(x) \tag{93}$$

In this way, we have dissociated the action on the variable x from the action of the variable t.

Let E be a Banach space of measurable functions on $\mathbb{R}^n$ satisfying $\| |f| \|_E \leq C_E \|f\|_E$. We see that $X_{E,\delta} = \{f \;/\; \sup_{t>0} t^{\delta/\alpha} \|f(t,.)\|_E < +\infty\}$ will be contained in $\mathcal{V}^{\alpha,\beta}$ if $(f,g) \mapsto I_{\alpha\gamma-\beta}(fg)$ is bounded from $E \times E$ to E and $(f,g) \mapsto I_{1-\gamma}(fg)$ is bounded on $X_t = \{f \;/\; |t|^{\delta/\alpha} f \in L^\infty\}$. We find that the maximal Banach space E we can associate (this way) to γ is $X_{E,\delta}$ with $E = \mathcal{V}^{\alpha\gamma-\beta}(\mathbb{R}^n) = \mathcal{M}(\dot{H}^{\alpha\gamma-\beta} \mapsto L^2)$ and $\gamma = 1 - \frac{\delta}{\alpha}$. Thus, we find that we can easily get cheap solutions when the initial value $\vec{u}_0$ belongs to (and is small in) X_0^d, with X_0 is the Besov space $X_0 = \dot{B}^{-\alpha+\beta+r}_{\mathcal{V}^r,\infty}$ with $\max(0, \frac{\alpha-2\beta}{2}) < r < \min(\alpha-\beta, \frac{n}{2})$ [18].

Due to the Fefferman–Phong inequality, we may replace the space $\mathcal{V}^r$ by a Morrey spce $\dot{M}^{s,n/r}$ with $2 < s \leq \frac{n}{r}$. The corresponding space X_0 will be a Besov-Morrey space $B^{-\frac{\alpha}{p}}_{\dot{M}^{s,q},\infty}$ (see Kozono and Yamazaki [17]) with Serrin's scaling relation $\frac{\alpha}{p} + \frac{n}{q} = \alpha - \beta$ (and with $2 < s \leq q$, $\frac{n}{\alpha-\beta} < q$ and, if $\alpha > 2\beta$, $q < \frac{2n}{\alpha-2\beta}$). If $s = q$, we find the classical Besov space $B^{-\frac{\alpha}{p}}_{q,\infty}$ (see Cannone [5]).

We have more precisely the following result :

Theorem 8 *Let $0 < \beta < \alpha \leq n + 2\beta$. Let $X^{\alpha,\beta}$ be the Banach space of distributions such that $1_{0<t}\; e^{-t(-\Delta)^{\alpha/2}} u_0 \in \mathcal{V}^{\alpha,\beta}$. Then :*

- $X^{\alpha,\beta} \subset \dot{B}^{\beta-\alpha}_{\infty,\infty}$
- *if $\beta > \alpha/2$, then*

$$\frac{1}{|t|^{1-\frac{\beta}{\alpha}}} \in \mathcal{V}^{\alpha,\beta} \tag{94}$$

so that $X^{\alpha,\beta} = \dot{B}^{\beta-\alpha}_{\infty,\infty}$.

- *if $\beta \leq \alpha/2$, there exists $u_0 \in \dot{B}^{\beta-\alpha}_{\infty,\infty}$ such that $u_0 \notin X^{\alpha,\beta}$. More precisely :*
  - *if $\beta < \alpha/2$, then there exists $u_0 \in \dot{B}^{\beta-\alpha}_{\infty,1}$ such that $u_0 \notin X^{\alpha,\beta}$.*
  - *if $\beta \leq \alpha/2$, then $\dot{B}^{\beta-\alpha+\frac{n}{q}}_{q,\infty} \subset X^{\alpha,\beta} \Leftrightarrow q < \frac{2n}{\alpha-2\beta}$.*

Proof : We already know that $X^{\alpha,\beta} \subset \dot{B}^{\beta-\alpha}_{\infty,\infty}$. If $\beta > \alpha/2$ and $2 < r < \frac{\alpha}{\alpha-\beta}$, we have

$$\iint_{\rho_\alpha(t-s,x-y)<R} \frac{1}{\left(|s|^{1-\frac{\beta}{\alpha}}\right)^r}\, ds\, dy \leq C \int_0^{R^\alpha} \frac{R^n}{\left(|s|^{1-\frac{\beta}{\alpha}}\right)^r}\, ds = C' R^{n+\alpha-r(\alpha-\beta)} \tag{95}$$

This inequality gives that $\frac{1}{|t|^{1-\frac{\beta}{\alpha}}} \in \dot{M}_\alpha^{r,\frac{n+\alpha}{\alpha-\beta}} \subset \mathcal{V}^{\alpha,\beta}$

We now consider the case $2\beta \leq \alpha$. We shall consider the cheap parabolic equation of Montgomery–Smith [24] :

$$\partial_t u + (-\Delta)^{\alpha/2} u = (-\Delta)^{\beta/2}(u^2) \tag{96}$$

and the associated bilinear operator

$$B_{\alpha,\beta}(u,v) = \int_0^t e^{-(t-s)(-\Delta)^{\alpha/2}} (-\Delta)^{\beta/2}(u(s,.)v(s,.))\; ds. \tag{97}$$

Let $\theta \in \mathcal{S}(\mathbb{R}^n)$ such that $1_{|\xi|<1} \leq \hat{\theta}(\xi) \leq 1_{|\xi|<2}$. For $\gamma \in \mathbb{R}$, we take $u_\gamma = 2\sum_{j\geq 3} \theta(x)\cos(2^j x_1) 2^{\gamma j}$. Then u_γ belongs to $\dot{B}^{-\gamma}_{q,\infty}$ for every $q \in [1,+\infty]$, and belongs to $\dot{B}^{\delta}_{\infty,1}$ for every $\delta < -\gamma$. Let $v_{\alpha,\gamma} = e^{-(t(-\Delta)^{\alpha/2}} u_\gamma$. If $B_{\alpha,\beta}(v_{\alpha,\gamma}, v_{\alpha,\gamma})$ is well defined, we check it agains a test function $\omega(t,x)$ which satisfies, in spatial Fourier variables,

$$1_{1/2<t<1} 1_{|\xi|<1} \leq \hat{\omega}(t,\xi) \tag{98}$$

For $|\eta| = \Omega(2^j)$, $|\xi| < 1$, $1/2 < t < 1$, we have

$$\begin{aligned} \int_0^t e^{-(t-s)|\xi|^\alpha - s|\eta|^\alpha - s|\xi-\eta|^\alpha}\; ds &\geq e^{-1} \int_0^t e^{-s|\eta|^\alpha - s|\xi-\eta|^\alpha}\; ds \\ &\geq e^{-1} \frac{1 - e^{-t(|\eta|^\alpha + |\xi-\eta|^\alpha)}}{|\xi-\eta|^\alpha + |\eta|^\alpha} \\ &\geq c_\alpha 2^{-\alpha j}. \end{aligned} \tag{99}$$

We thus get (with $\epsilon_1 = (1, 0, \ldots, 0)$)

$$\begin{aligned} &(2\pi)^n \langle B_{\alpha,\beta}(v_{\alpha,\gamma}, v_{\alpha,\gamma}) | \omega \rangle \geq \\ c_\alpha \int_{1/2}^{1} \int_{|\xi|<1} |\xi|^\beta \sum_j 2^{(2\gamma-\alpha)j} &\int \hat{\theta}(\xi - \eta - 2^j \epsilon_1) \hat{\theta}(\eta + 2^j \epsilon_1) \, d\xi \, dt \\ &\geq c'_\alpha \sum_{j=3}^{+\infty} 2^{j(2\gamma-\alpha)} \end{aligned} \tag{100}$$

with $c'_\alpha > 0$. Thus, $B_{\alpha,\beta}(v_{\alpha,\gamma}, v_{\alpha,\gamma})$ cannot be well defined for $2\gamma \geq \alpha$.

Thus, $u_{\alpha/2} \notin X^{\alpha,\beta}$. But we know that $u_{\alpha/2} \in \dot{B}^{\beta-\alpha}_{\infty,1}$ if $\beta - \alpha < -\alpha/2$, i.e. $\beta < \alpha/2$. Similatly, if $\beta \leq \alpha/2$ and $q = \frac{2n}{\alpha - 2\beta}$, we know that $u_{\alpha/2} \in \dot{B}^{-\frac{\alpha}{2}}_{q,\infty} = \dot{B}^{\beta-\alpha+\frac{n}{q}}_{q,\infty}$. Theorem 8 is thus proved. ◇

Remark : In this paper, we deal only with critical spaces and global existence. But it is easy to check that the same example of the cheap equation and of the initial value $u_{\alpha/2}$ gives that there is no local existence results for the subcritical spaces $\dot{B}^{-\delta}_{\infty,\infty}$ with $\alpha/2 \leq \delta < \alpha - \beta$.

9 The case $\alpha = 2\beta$.

We have seen that for $\beta > \alpha/2$ we had $\dot{B}^{\beta-\alpha}_{\infty,\infty} \subset X^{\alpha,\beta}$, so that the Cauchy problem for our general parabolic equation with a small initial value in $(\dot{B}^{\beta-\alpha}_{\infty,\infty})^d$ will have a solution. For $\beta > \alpha/2$, we found an example $u_{\alpha/2} \in \dot{B}^{\beta-\alpha}_{\infty,1}$ so that, for every $\lambda > 0$, the Cauchy problem for the cheap equation with the initial value $\lambda u_{\alpha/2}$ will have no solution.

In the limit case $\beta = \alpha/2$, the counter-example $u_{\alpha/2}$ belongs to $\dot{B}^{-\alpha/2}_{\infty,\infty}$, so that $\dot{B}^{-\alpha/2}_{\infty,\infty}$ is not included in $X^{\alpha,\alpha/2}$. However, the Cauchy problem for the general parabolic equation with a small data in $(\dot{B}^{-\alpha/2}_{\infty,1})^d$ will have a global solution. As a matter of fact, the Koch and Tataru theorem [16] gives that this is true for a small initial value in $(BMO^{-\alpha/2})^d$ where we have $\dot{B}^{-\alpha/2}_{\infty,1} \subset \dot{B}^{-\alpha/2}_{\infty,2} \subset BMO^{-\alpha/2} = (-\Delta)^{\alpha/4} BMO = \dot{F}^{-\alpha/2}_{\infty,2} \subset \dot{B}^{-\alpha/2}_{\infty,\infty}$).

We don't detail the proof here, as it is exactly the same one as for the Koch and Tataru theorem (see [18] for details). The path space where to use the fixed-point theorem is the space of functions $u(t, x)$ which satisfy $\sup_{t>0} t^{1/2} \|u(t, .)\|_\infty < +\infty$ and

$$\sup_{t>0, x \in \mathbb{R}^n} t^{-\frac{n}{\alpha}} \int_0^t \int_{B(x, t^{\frac{1}{\alpha}})} |u(s, y)|^2 \, ds \, dy < +\infty \tag{101}$$

Note that the proof involves an integration by parts [using the fact that $(-\Delta)^{\alpha/2}e^{-t(-\Delta)^{\alpha/2}}f = -\partial_t(e^{-t(-\Delta)^{\alpha/2}}f)$, see Lemma 16.2 in [18]]. Thus, the proof does not involve domination by a positive kernel, and $BMO^{-\alpha/2}$ is not a subspace of $X^{\alpha,\alpha/2}$. But we have obviously (due to scaling invariance and local square integrability in $\mathcal{V}^{\alpha,\alpha/2}$) the embedding $X^{\alpha,\alpha/2} \subset BMO^{-\alpha/2}$.

10 Persistency.

When $\vec{u}_0$ is small in $(X^{\alpha,\beta})^d$ (or, when $\alpha = 2\beta$, in $(BMO^{-\alpha/2})^d$), we know that a solution $\vec{u}$ may be constructed through the iteration algorithm :

$$\vec{U}_0 = e^{-t(-\Delta)^{\alpha/2}}\vec{u}_0 \text{ and } \vec{U}_{k+1} = \vec{U}_0 + \int_0^t e^{-(t-s)(-\Delta)^{\alpha/2}}\sigma(D)(\vec{U}_k \otimes \vec{U}_k)\, ds. \quad (102)$$

and that we have

$$\sup_{0<t} t^{1-\frac{\beta}{\alpha}} \|\vec{U}_0(t,.)\|_\infty + \sum_{k=0}^{+\infty} \sup_{0<t} t^{1-\frac{\beta}{\alpha}} \|\vec{U}_{k+1}(t,.) - \vec{U}_k(t,.)\|_\infty < +\infty. \quad (103)$$

This will allow us to use the persistency theory developed in [18]. Let us recall first the definition of a shift-invariant Banach space of local measures :

Definition 3 *A) A shift-invariant Banach space of test functions is a Banach space E so that we have the continuous embeddings $\mathcal{D}(\mathbb{R}^n) \subset E \subset \mathcal{D}'(\mathbb{R}^n)$ and so that:*

- *for all $x_0 \in \mathbb{R}^d$ and for all $f \in E$, $f(x - x_0) \in E$ and $\|f\|_E = \|f(x - x_0)\|_E$.*
- *for all $\lambda > 0$ there exists $C_\lambda > 0$ so that for all $f \in E$ $f(\lambda x) \in E$ and $\|f(\lambda x)\|_E \leq C_\lambda \|f\|_E$.*
- *$\mathcal{D}(\mathbb{R}^d)$ is dense in E*

B) A shift-invariant Banach space of distributions is a Banach space E, which is the topological dual of a shift-invariant Banach space of test functions $E^{(*)}$. The space $E^{(0)}$ of smooth elements of E is defined as the closure of $\mathcal{D}(\mathbb{R}^d)$ in E.
C) A shift-invariant Banach space of local measures is a shift-invariant Banach space of distributions E so that for all $f \in E$ and all $g \in \mathcal{S}(\mathbb{R}^d)$ we have $fg \in E$ and $\|fg\|_E \leq C_E \|f\|_E \|g\|_\infty$, where C_E is a positive constant (which depends neither on f nor on g).

An important property of shift–invariant Banach spaces E of distributions or of test functions is that convolution is a bounded bilinear operator from $L^1 \times E$ to E : $\|f * g\|_E \leq \|f\|_1 \|g\|_E$.

We measure regularity with semi–norms $\|f\|_{\dot H^\rho_E} = \|(-\Delta)^{\rho/2} f\|_E$, or $\|f\|_{\dot B^\rho_{E,q}} = \left(\sum_{j\in\mathbb{Z}} 2^{j\rho q} \|\Delta_j f\|_E^q\right)^{1/q}$. Those are only semi–norms, but we shall work in spaces $L^\infty \cap \dot H^\rho$, or $L^\infty \cap \dot B^\rho_{E,q}$, so that we don't bother on the kernel of the semi–norms.

The presistency theory then tells us the following :

Theorem 9 *Let $\vec u_0$ be small enough in $(X^{\alpha,\beta})^d$ (or, when $\alpha = 2\beta$, in $(BMO^{-\alpha/2})^d$) to grant that*

$$\sup_{0<t} t^{1-\frac{\beta}{\alpha}} \|\vec U_0(t,.)\|_\infty + \sum_{k=0}^{+\infty} \sup_{0<t} t^{1-\frac{\beta}{\alpha}} \|\vec U_{k+1}(t,.) - \vec U_k(t,.)\|_\infty < +\infty. \tag{104}$$

and

$$\sup_{0<t} \|\vec U_0(t,.)\|_{\dot B^{\beta-\alpha}_{\infty,\infty}} + \sum_{k=0}^{+\infty} \sup_{0<t} \|\vec U_{k+1}(t,.) - \vec U_k(t,.)\|_{\dot B^{\beta-\alpha}_{\infty,\infty}} < +\infty. \tag{105}$$

Let F be a shift-invariant Banach space of local measures.

- *If moreover $\vec u_0 \in F^d$, then the limit $\vec u$ of $\vec U_k$ satisfies $\vec u \in L^\infty((0,+\infty), F^d)$.*
- *Let E be a space of regular distributions over F: for some positive ρ and for some $q \in [1,+\infty]$, $E = \dot H^\rho_F$ or $E = \dot B^\rho_{F,q}$ (with $1 \leq q \leq \infty$). If $\vec u_0 \in E^d$ then $\vec u \in L^\infty((0,\infty), E^d)$*

Proof : If $\vec u_0 \in F^d$, then $\vec U_0 \in L^\infty((0,+\infty), F^d)$. We then write, for $\vec W_k = \vec U_k - \vec U_{k-1}$ and $\alpha_k = \sup_{t>0} t^{1-\frac{\beta}{\alpha}} \|\vec W_k(t,.)\|_\infty$, :

$$\begin{aligned} \|\vec W_{k+1}(t,.)\|_F &\leq \int_0^t \frac{C}{|t-s|^{\frac{\beta}{\alpha}}} \|\vec U_k(s,.) \otimes \vec W_k(s,.) + \vec W_k(s,.) \otimes \vec U_{k-1}(s,.)\|_F \, ds \\ &\leq (\|\vec U_k\|_{L^\infty_t F} + \|\vec U_{k-1}\|_{L^\infty_t F}) \alpha_k \int_0^t \frac{C}{|t-s|^{\frac{\beta}{\alpha}}} \frac{1}{|s|^{1-\frac{\beta}{\alpha}}} \, ds \\ &= C'(\|\vec U_k\|_{L^\infty_t F} + \|\vec U_{k-1}\|_{L^\infty_t F}) \alpha_k \end{aligned} \tag{106}$$

If $A_k = \sum_{j=0}^k \|\vec W_j\|_{L^\infty_t F}$, we have $\|\vec u\|_{L^\infty_t F} \leq \sup_{k\in\mathbb{R}^n} A_j$. Moreover, we have

$$A_{k+1} = A_k + \|\vec W_{k+1}\|_{L^\infty_t F} \leq A_k(1 + 2C'\alpha_k) \tag{107}$$

so that $\vec{u} \in L^\infty((0,+\infty), F^d)$ with $\|\vec{u}\|_{L^\infty_t F} \le \|\vec{u}_0\|_F \prod_{k=0}^\infty (1+2C'\alpha_k)$

We now consider the case when $\vec{u}_0 \in E^d$. We find that $\vec{U}_0 \in L^\infty((0,+\infty), (\dot B^\rho_{F,\infty})^d)$. We write $\vec{W}_k = \vec{U}_k - \vec{U}_{k-1}$, $\alpha_k = \sup_{t>0} t^{1-\frac{\beta}{\alpha}} \|\vec{W}_k(t,.)\|_\infty$, $\Gamma = \sum_{k\in\mathbb{N}} \|\vec{W}_k\|_{L^\infty \dot B^{\beta-\alpha}_{\infty,\infty}}$ and $B_k = \sum_{j=0}^k \|\vec{W}_j\|_{L^\infty_t \dot B^\rho_{F,\infty}}$.

We begin by estimating fg when $f, g \in L^\infty \cap \dot B^{\beta-\alpha}_{\infty,\infty} \cap \dot B^\rho_{F,\infty}$. Using the Littlewood–Paley decomposition $f = \sum_{j\in\mathbb{Z}} \Delta_j f = S_k f + \sum_{j\ge k} \Delta_j f$ (see [18]), we write $fg = u + v$, where $u = \sum_k \sum_{j\le k+3} \Delta_j f \Delta_k g = \sum_{k\in\mathbb{Z}} S_{k+4} f \Delta_k g$ and $v = \sum_k \sum_{j\ge k+4} \Delta_j f \Delta_k g = \sum_j \Delta_j f S_{j-3} g$. We have$\|\Delta_l(S_{k+4} f \Delta_k g)\|_F \le C\|f\|_\infty \|\Delta_k g\|_F \le C\|f\|_\infty \|g\|_{\dot B^\rho_{F,\infty}} 2^{-k\rho}$ if $k \ge l-6$, and $= 0$ if $k < l-6$. Hence $u \in \dot B^\rho_{F,\infty}$ and

$$\|u\|_{\dot B^\rho_{F,\infty}} \le C\|f\|_\infty \|g\|_{\dot B^\rho_{F,\infty}}. \tag{108}$$

On the other hand, when $k \le j-4$, we have $\Delta_l(\Delta_j f \Delta_k g)\|_F = 0$ if $|l-j| \ge 3$; when $|l-j| \le 2$, we write $\|\Delta_l(\Delta_j f \Delta_k g)\|_F \le C 2^{-k\rho} \|g\|_{\dot B^\rho_{F,\infty}} \|f\|_\infty$ and $\|\Delta_l(\Delta_k f \Delta_j g)\|_F \le 2^{k(\alpha-\beta)} \|g\|_{\dot B^{\beta-\alpha}_{\infty,\infty}} 2^{-j\rho} \|f\|_{\dot B^\rho_{F,\infty}}$ We then fix λ such that $\frac{\rho}{\rho+\alpha-\beta} < \lambda < 1$, and we find that

$$\|\Delta_l(\Delta_j f S_{j-3} g)\|_F \le C 2^{j(-\rho+\lambda(\alpha-\beta))} (\|f\|_{\dot B^\rho_{F,\infty}} \|g\|_{\dot B^{\beta-\alpha}_{\infty,\infty}})^\lambda (\|g\|_{\dot B^\rho_{F,\infty}} \|f\|_\infty)^{1-\lambda} \tag{109}$$

and thus

$$\| v\|_{\dot B^{\rho-\lambda(\alpha-\beta)}_{\infty,\infty}} \le C(\|f\|_{\dot B^\rho_{F,\infty}} \|g\|_{\dot B^{\beta-\alpha}_{\infty,\infty}})^\lambda (\|g\|_{\dot B^\rho_{F,\infty}} \|f\|_\infty)^{1-\lambda} \tag{110}$$

The second step is to check that $e^{-(t-s)(-\Delta)^{\alpha/2}} \sigma(D)$ maps $(\dot B^\delta_{F,\infty})^{d\times d}$ to $(\dot B^\rho_{F,1})^d$ for $\delta < \rho + \beta$:

$$\|e^{-(t-s)(-\Delta)^{\alpha/2}} \sigma(D) f\|_{\dot B^\rho_{F,1}} \le C(t-s)^{-\frac{\rho+\beta-\delta}{\alpha}} \|f\|_{\dot B^\delta_{F,\infty}} \tag{111}$$

Combining those estimates, we find that

$$\begin{aligned}\|\vec{W}_{k+1}(t,.)\|_{\dot B^\rho_{F,1}} \le \alpha_k B_k \int_0^t \frac{C}{|t-s|^{\frac{\beta}{\alpha}}} \frac{1}{s^{1-\frac{\beta}{\alpha}}}\, ds \\ + (\alpha_k B_k)^\lambda (\Gamma \|\vec{W}_k(t,.)\|_{\dot B^\rho_{F,1}})^{1-\lambda} \int_0^t \frac{C}{|t-s|^{\frac{\beta}{\alpha}+(1-\lambda)(1-\frac{\beta}{\alpha})}} \frac{1}{s^{\lambda(1-\frac{\beta}{\alpha})}}\, ds\end{aligned} \tag{112}$$

We take λ close enough to 1 to ensure that

$$\lambda(1-\frac{\beta}{\alpha}) < 1 \text{ and } \frac{\beta}{\alpha} + (1-\lambda)(1-\frac{\beta}{\alpha}) < 1. \tag{113}$$

We thus get

$$\|\vec{W}_{k+1}(t,.)\|_{\dot{B}^{\rho}_{F,1}} \leq C\alpha_k B_k + C(\alpha_k B_k)^{\lambda}(\Gamma\|\vec{W}_k(t,.)\|_{\dot{B}^{\rho}_{F,1}})^{1-\lambda}. \tag{114}$$

We find

$$\|\vec{W}_{k+1}(t,.)\|_{\dot{B}^{\rho}_{F,1}} \leq \delta_k B_k + \frac{1}{2}\|\vec{W}_k(t,.)\|_{\dot{B}^{\rho}_{F,1}} \tag{115}$$

with $\delta_k = C\alpha_k(1+\lambda(2(1-\lambda)\Gamma)^{\frac{1-\lambda}{\lambda}})$. For $1 \leq p \leq k$, we have as well $\|\vec{W}_p(t,.)\|_{\dot{B}^{\rho}_{F,1}} \leq \delta_{p-1}B_k + \frac{1}{2}\|\vec{W}_{p-1}(t,.)\|_{\dot{B}^{\rho}_{F,1}}$, while $\|\vec{W}_0(t,.)\|_{\dot{B}^{\rho}_{F,1}} \leq \delta_{-1}B_k$ if we take $\delta_{-1} = 1$. This gives

$$\|\vec{W}_{k+1}(t,.)\|_{\dot{B}^{\rho}_{F,1}} \leq B_k(\sum_{j=-1}^{k}\delta_j 2^{j-k}) \tag{116}$$

so that

$$B_{k+1} \leq B_k(1+\sum_{j=-1}^{k}\delta_j 2^{j-k}) \tag{117}$$

and finally

$$\sup_{k\in\mathbb{N}} B_k \leq B_0 \prod_{k=0}^{+\infty}(1+\sum_{j=-1}^{k}\delta_j 2^{j-k}) < +\infty \tag{118}$$

The theorem is proved : for $E = \dot{H}^{\sigma}_F$ or $\dot{B}^{\rho}_{F,q}$, we have

$$\|\vec{u}\|_{L^\infty E} \leq \|\vec{u}_0\|_E + \sum_{k=1}^{\infty}\|\vec{W}_k\|_{L^\infty \dot{B}^{\rho}_{\infty,1}} < +\infty \tag{119}$$

and we conclude since $\dot{B}^{\rho}_{\infty,1} \subset E$. ◇

11 A Triebel-space approach of cheap solutions.

Recall that $X^{\alpha,\beta}$ is defined by $u_0 \in X^{\alpha,\beta} \Leftrightarrow 1_{t>0}e^{-t(-\Delta)^{\alpha/2}}u_0 \in \mathcal{V}^{\alpha,\beta}$. In section 8, we tried to give an approximation of $X^{\alpha,\beta}$ by Besov spaces. Another way of approximating $X^{\alpha,\beta}$ is to approach $\mathcal{V}^{\alpha,\beta}$ with Morrey spaces, using the Fefferman–Phong inequality.

We thus define $\mathcal{F}^{\alpha,\beta}_p$ for $2 < p \leq \frac{n+\alpha}{\alpha-\beta}$ by :

$$u_0 \in \mathcal{F}^{\alpha,\beta}_p \Leftrightarrow 1_{t>0}e^{-t(-\Delta)^{\alpha/2}}u_0 \in \dot{M}^{p,\frac{n+\alpha}{\alpha-\beta}}_{\alpha} \tag{120}$$

We have of course (for $2 < p \leq \frac{n+\alpha}{\alpha-\beta}$)

$$\mathcal{F}_p^{\alpha,\beta} \subset X^{\alpha,\beta} \subset \dot{B}^{\beta-\alpha}_{\infty,\infty}. \tag{121}$$

Assume now that $p\frac{\alpha-\beta}{\beta} > 1$. For $R > 0$ and $x_0 \in \mathbb{R}^n$, we find that

$$\begin{aligned}\int_{B(x_0,R)} \int_0^{+\infty} |e^{-t(-\Delta)^{\alpha/2}} u_0|^p \, dt \, dy \leq & \iint_{\rho_\alpha(t-0,y-x_0)<R} |1_{t>0} e^{-t(-\Delta)^{\alpha/2}} u_0|^p \, dt \, dy \\ & + \int_{B(x_0,y)} \int_{R^\alpha}^{+\infty} |e^{-t(-\Delta)^{\alpha/2}} u_0|^p \, dt \, dy \\ & \leq C \|u_0\|^p_{\mathcal{F}_p^{\alpha,\beta}} \, R^{n+\alpha-p(\alpha-\beta)} \\ & + C\|u_0\|_{\dot{B}^{\beta-\alpha}_{\infty,\infty}} R^n R^{\alpha(1-p\frac{\alpha-\beta}{\alpha})}\end{aligned} \tag{122}$$

Thus, we find that $\left(\int_0^{+\infty} |e^{-t(-\Delta)^{\alpha/2}} u_0|^p \, dt\right)^{1/p} \in \dot{M}^{p,q}(\mathbb{R}^n)$, where q satisfies the Serrin scaling relation $\frac{\alpha}{p} + \frac{n}{q} = \alpha - \beta$. We thus see that $\mathcal{F}_p^{\alpha,\beta}$ is aTriebel–Lizorkin–Morrey space, as studied by Sickel, Yang and Yuan [26] :

Theorem 10 *For $2 < p < \frac{n+\alpha}{\alpha-\beta}$ such that $p\frac{\alpha-\beta}{\beta} > 1$, the space $\mathcal{F}_p^{\alpha,\beta}$ is equal to the homogeneous Triebel–Lizorkin–Morrey space $\dot{F}_{p,p}^{-\frac{\alpha}{p},\frac{1}{p}-\frac{1}{q}}$.*

12 Examples

12.1 The Navier–Stokes equations.

The Navier–Stokes equations are given on $(0, +\infty) \times \mathbb{R}^3$ by

$$\begin{cases} \partial_t \vec{u} + (\vec{u}.\vec{\nabla})\vec{u} = \Delta\vec{u} - \vec{\nabla}p \\ \qquad\qquad \text{div } \vec{u} = 0 \end{cases} \tag{123}$$

Using the Leray projection operator $\mathbb{P}$ on divergence-free vector fields and the fact that $\vec{u}$ is divergence free, we get rid of the pressure (on the assumption that p is small at infinity) and get

$$\partial_t \vec{u} = \Delta\vec{u} - \mathbb{P} \text{ div } (\vec{u} \otimes \vec{u}) = 0 \tag{124}$$

This is a system of equations analogous to (2) with $\alpha = 2$ and $\beta = 1$. Since 2001, from the Koch and Tataru theorem [16], we know that we may find a global solution as soon as the initial value $\vec{u}_0$ is small enough in BMO^{-1}.

Initially, in 1964 [10], the proof of existence of global solutions has been given for an initial value in $H^s(\mathbb{R}^3)$ with $s \geq 1/2$ and with a small norm in $\dot{H}^{1/2}(\mathbb{R}^3)$. It is easy to see that $\dot{H}^{1/2} \subset X^{2,1}$ so that the existence of a global solution in $L^\infty_t H^s$ is then a combination of Theorems 7 and 9.

Later, in 1984 [15], Kato proved existence of global solutions in $L^\infty_t L^3$ for an initial value d with a small norm in L^3. Again, this can be proved through a combination of Theorems 7 and 9, as $L^3 \subset X^{2,1}$.

Then, in 1995 [5], Cannone considered the case of an initial value in L^3, with a small norm in $\dot{B}^{-1+\frac{3}{q}}_{q,\infty}$ with $3 < q < +\infty$ and obtained existence of a global solutions in $L^\infty_t L^3$. Again, this can be proved through a combination of Theorems 7 and 9, as $\dot{B}^{-1+\frac{3}{q}}_{q,\infty} \subset X^{2,1}$.

Let us remark that ill-posedness in the critical Besov space $\dot{B}^{-1}_{\infty,\infty}$ has been established in 2008 by Bourgain and Pavlović [4], following the examples given by Montgomery–Smith for the cheap equation [24].

12.2 The modified Navier–Stokes equations.

The diffusion term in the Navier–Stokes equations has been modified in some studies by a fractional diffusion :

$$\begin{cases} \partial_t \vec{u} + (\vec{u}.\vec{\nabla})\vec{u} = -(-\Delta)^{\alpha/2}\vec{u} - \vec{\nabla} p \\ \text{div } \vec{u} = 0 \end{cases} \tag{125}$$

Initially, α was taken larger than 2 (it is the hyperdiffusive case). Indeed, when $\alpha > 5/2$, the problem is locally well posed in L^2, and, using the energy inequality that ensures that the norm in L^2 stays bounded, local existence is turned into global existence [21]. More recently, the sase $1 < \alpha < 2$ has been considered, due to the increase use of α–stable process in non-local diffusion models.

Using again the Leray projection operator $\mathbb{P}$, we get the system

$$\partial_t \vec{u} = -(-\Delta)^{\alpha/2}\vec{u} - \mathbb{P} \text{ div } (\vec{u} \otimes \vec{u}) = 0 \tag{126}$$

This is a system of equations analogous to (2) with $\alpha > 1$ and $\beta = 1$.

When $1 < \alpha < 2$, we know from Theorems 7 and 8 that we may find a global solution as soon as the initial value $\vec{u}_0$ is small enough in $\dot{B}^{1-\alpha}_{\infty,\infty}$ (this is the theorem of Yu and Zhai [28]).

When $\alpha > 2$, in accordance with Theorem 8 and the remark we made after the Theorem, Cheskidov and Shvydkoy [6] have shown illposedness in $\dot{B}^{\gamma}_{-\infty,\infty}$ for $1 - \alpha \leq \gamma \leq -\alpha/2$.

12.3 The subcritical quasi-geostrophic equation

The subcritical quasi-geostrophic equation is given by the system

$$\begin{cases} \partial_t \theta + (\vec{u}.\vec{\nabla})\theta = -(-\Delta)^{\alpha/2}\theta \\ (u_1, u_2) = (-\frac{\partial_2}{\sqrt{-\Delta}}\theta, \frac{\partial_1}{\sqrt{-\Delta}}\theta) \end{cases} \tag{127}$$

where $1 < \alpha < 2$.

If we use the unknowns (θ, u_1, u_2), we get the system

$$\begin{cases} \partial_t \theta = -(-\Delta)^{\alpha/2}\theta - \text{ div } (\theta\vec{u}) \\ \partial_t u_1 = -(-\Delta)^{\alpha/2}u_1 + \frac{1}{\sqrt{-\Delta}}\partial_2 \text{ div } (\theta\vec{u}) \\ \partial_t u_2 = -(-\Delta)^{\alpha/2}u_2 - \frac{1}{\sqrt{-\Delta}}\partial_1 \text{ div } (\theta\vec{u}) \end{cases} \tag{128}$$

This is a system of equations analogous to (2) with $1 < \alpha < 2$ and $\beta = 1$. We know from Theorems 7 and 8 that we may find a global solution as soon as the initial value θ_0 is small enough in $\dot{B}^{1-\alpha}_{\infty,\infty}$ (this is the theorem of May and Zahrouni[22]).

In particular, when $\theta_0 \in L^{\frac{2}{\alpha-1}} \subset \dot{B}^{1-\alpha}_{\infty,\infty}$ and is small in $\dot{B}^{1-\alpha}_{\infty,\infty}$, we know that the solution θ satisfies $\theta \in L^\infty L^{\frac{2}{\alpha-1}}$. If $\theta_0 \in L^q$ with $\frac{2}{\alpha-1} < q < +\infty$, we have local existence in $L^\infty L^q$; moreover θ satisfies a maximum principle : $\|\theta(t,.)\|_q \leq \|\theta_0\|_q$, and this implies that local existence is turned into global existence [27]

12.4 The parabolic-elliptic Keller–Segel system

The parabolic-elliptic Keller–Segel system is given on $(0, +\infty) \times \mathbb{R}^n$ by

$$\begin{cases} \partial_t u = \Delta u - \text{ div } (u\vec{\nabla}\chi) \\ -\Delta\chi = u \end{cases} \tag{129}$$

If we use the unknowns $\vec{v} = \vec{\nabla}\chi = -\frac{1}{\Delta}\vec{\nabla}u$, we get the system

$$\partial_t \vec{v} = \Delta\vec{v} + \sum_{i=1}^n \frac{1}{\Delta}\vec{\nabla} \text{ div } \partial_i(v_i\vec{v}) - \frac{1}{2}\vec{\nabla}(\sum_{i=1}^n v_i^2) \tag{130}$$

This is a system of equations analogous to (2) with $\alpha = 2$ and $\beta = 1$. We thus know that we may find a global solution as soon as the initial value $\vec{v}_0$ is small enough in BMO^{-1}, i.e. u_0 is small enough in BMO^{-2}. This estimate seems to be new : in [13], the case $u_0 \in \dot{B}^{-2+\frac{2}{q}}_{q,\infty}$ is discussed.

Let us assume that $u_0 \in L^{n/2} \cap L^1$ (and $n \geq 2$), with the norm of u_0 is small enough in BMO^{-2} (remark that $L^{d/2} \subset BMO^{-2}$). Then we know from Theorem 9 that the solution $\vec{v}$ will belong to $L^\infty \dot{H}^1_{L^{d/2}} \cap L^\infty \dot{H}^1_{L^1}$, and that $\vec{w} = \vec{v} - e^{t\Delta}\vec{v}_0 \in L^\infty \dot{B}^1_{d/2,1} \cap L^\infty \dot{B}^1_{1,1}$. Writing

$$u = e^{t\Delta} u_0 + \text{ div } \vec{w}, \tag{131}$$

we find that $u \in L^\infty L^{d/2} \cap L^\infty L^1$: this is the theorem of Corrias, Perthame and Zaag [8].

A final remark is that one usually deals with positive solutions (as u represents a density of cells). We have the inequalities

$$\|u_0\|_{\dot{B}^{-2}_{\infty,\infty}} \leq C\|u_0\|_{BMO^{-2}} \leq C'\|u_0\|_{\dot{M}^{1,d/2}} \tag{132}$$

when the space $\dot{M}^{1,d/2}$ is the space of locally bounded (signed) measures μ such that : $\sup_{x_0 \in \mathbb{R}^n, R>0} R^{2-d} \int_{B(x_0,R)} d|\mu(y)| < +\infty$. When u_0 is a non-negative distribution (i.e. a non-negative locally bounded measure), we have the reverse inequality

$$\|u_0\|_{\dot{M}^{1,d/2}} \leq C''\|u_0\|_{\dot{B}^{-2}_{\infty,\infty}} \tag{133}$$

(see [20]). Thus, the critical norm to be controlled is indeed the norm in the Morrey space $\dot{M}^{1,d/2}$.

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