On some classes of time-periodic solutions for the Navier–Stokes equations in the whole space.

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Abstract

For the 3D Navier–Stokes problem on the whole space, we study existence, regularity and stability of time-periodic solutions in Lebesgue, Lorentz or Sobolev spaces, when the periodic forcing belongs to critical classes of forces.

Keywords : Navier–Stokes equations, periodic solutions, maximal functions

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Introduction.

In this paper, we describe some classes of time-periodic forces that generate time-periodic solutions of the incompressible Navier–Stokes in the whole space and we discuss the stability of the solutions.

More precisely, we consider the equations on $(0,+\infty)\times\mathbb{R}^3$

$$\left\{\begin{array}{r}\partial_t\vec u+\ \mathrm{div}\ (\vec u\otimes\vec u)=\Delta\vec u+\vec f_{\mathrm{per}}-\vec\nabla p\\ \mathrm{div}\ \vec u=0\\ \vec u(0,.)=\vec u_0\end{array}\right. \tag{1}$$

where $\vec f_{\mathrm{per}}$ is time-periodic :

$$\vec f_{\mathrm{per}}(t+T,x)=\vec f_{\mathrm{per}}(t,x)$$

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and where the pressure p is determined by the Leray projection :

$$\vec{\nabla}p = \vec{\nabla}\frac{1}{\Delta} \text{ div } (\vec{f}_{\text{per}} - \text{ div } (\vec{u} \otimes \vec{u})).$$

We shall introduce functional spaces $\mathbb{W}, \mathbb{X}, \mathbb{Y}$ on $(0, +\infty) \times \mathbb{R}^3$ and $\mathbb{U}, \mathbb{V}$ on $\mathbb{R}^3$ such that :

- if $\|\vec{f}_{\text{per}}\|_{\mathbb{X}}$ is small enough, then there exists a $\vec{U}_0$ such that the initial value problem (1) with initial value $\vec{U}_0$ has a time-periodic solution $\vec{U}_{\text{per}} \in \mathbb{Y}$
- if moreover $\|\vec{u}_0\|_{\mathbb{U}}$ is small enough, the initial value problem (1) with initial value $\vec{u}_0$ has a global solution $\vec{u} \in \mathbb{W}$ and

$$\lim_{t \to +\infty} \|\vec{u}(t, .) - \vec{U}_{\text{per}}(t, .)\|_{\mathbb{V}} = 0.$$

A typical example of such spaces will be $\vec{f}_{\text{per}} = \text{div } F_{\text{per}}$ with a periodic tensor $F_{\text{per}} \in L^\infty_t L^{3/2,\infty}$ (where the Lorentz space $L^{3/2,\infty}$ is equal to the weak-$L^{3/2}$ space of Marcinkiewicz) and $\mathbb{Y} = L^\infty L^{3,\infty}$, $\mathbb{U} = L^{3,\infty}$, $\mathbb{W} = L^\infty_t L^{3/2,\infty}$ and $\mathbb{V} = L^{6,\infty}$. This case has been studied by Yamazaki in 2000 [19].

The study of time-periodic Navier-Stokes problem is now ancient, since the works of Serrin on bounded domains [18] and of Maremonti [12] on the whole space. Maremonti constructed the periodic solution $\vec{U}_{\text{per}}(t, .) = \vec{U}_{\text{per}}(t + NT, .)$ as the asymptotic limit of $\vec{u}(t + NT, .)$ of the Cauchy initial value problem for a small arbitrary initial value $\vec{u}_0$. Kozono and Nakao [10] used another construction, based on the formalism of mild solutions developed by Kato and Fujita [4, 7, 8] for the study of the Cauchy initial value problem. Then, Yamazaki [19] proved that the Lorentz space $L^\infty_t L^{3,\infty}$ was well fit to the search of periodic solutions.

Here, we shall provide a variation on Yamazaki's results, combined with the method of Kyed [11] based on the Fourier expansion in time variable

1 The Stokes equation

The first step is to describe eternal solutions of the linear heat equation in $L^\infty_t \dot{B}^{-1}_{\infty,\infty}$. This space is adapted to the symmetries of the Navier–Stokes equations :

- space-translation invariance :
  if $(\vec{u}, \vec{f}, p)$ satisfy

$$\partial_t \vec{u} + \text{ div } (\vec{u} \otimes \vec{u}) = \Delta \vec{u} + \vec{f}_{\text{per}} - \vec{\nabla}p \tag{2}$$

then so do $(\vec{u}(t, x - x_0), \vec{f}(t, x - x_0), p(t, x - x_0))$ for any $x_0 \in \mathbb{R}^3$

- time-translation invariance :
  if $(\vec{u}, \vec{f}, p)$ satisfy (2), then so do $(\vec{u}(t - t_0, x), \vec{f}(t - t_0, x), p(t - t_0, x))$ for any $t_0 \in \mathbb{R}$

- scale invariance :
  if $(\vec{u}, \vec{f}, p)$ satisfy (2), then so do $(\lambda\vec{u}(\lambda^2 t, \lambda x), \lambda^3 \vec{f}(\lambda^2 t, \lambda x), \lambda^2 p(\lambda^2 t, \lambda x))$ for any $\lambda > 0$

If we want to keep those invariances for a space of solutions $\mathbb{W}$ for $\vec{u}$, we may look for spaces $\mathbb{W} = L^p_t \mathbb{Z}_x$ (or $L^{p,\infty}_t \mathbb{Z}_x$, when $1 < p < \infty$) where the norm of $\mathbb{Z}$ is invariant through space translations ($\|u(x - x_0)\|_{\mathbb{Z}} = \|u\|_{\mathbb{Z}}$) and through scaling ($\lambda^{1-\frac{2}{p}} \|u(\lambda x)\|_{\mathbb{Z}} = \|u\|_{\mathbb{Z}}$). If $\mathbb{Z}$ is continuously embedded into $\mathcal{S}'$ and if a distribution u belongs to $\mathbb{Z}$, then we have, defining W as the Gaussian which generates the heat kernel,

$$e^{t\Delta} u(t, x) = \langle \frac{1}{t^{3/2}} W(\frac{y}{\sqrt{t}}) | u(y + x) \rangle_{\mathcal{S}', \mathcal{S}} = \langle W(y) | u(\sqrt{t}(y + \frac{x}{\sqrt{t}})) \rangle_{\mathcal{S}', \mathcal{S}}$$

and thus

$$\|e^{t\Delta} u\|_{L^\infty(dx)} \leq C \|u\|_{\mathbb{Z}} t^{-\frac{1}{2} + \frac{1}{p}}$$

If $p > 2$, this gives that $\mathbb{Z}$ is embedded into the homogeneous Besov space $\dot{B}^{-1+\frac{2}{p}}_{\infty,\infty}$.

Similarly, if we want to keep those invariances for a space of forces $\mathbb{F}$ for $\vec{f}$, we may look for spaces $\mathbb{F} = L^p_t \mathbb{G}_x$ (or $L^{p,\infty}_t \mathbb{G}_x$, when $1 < p < \infty$) where the norm of $\mathbb{G}$ is invariant through space translations ($\|f(x - x_0)\|_{\mathbb{G}} = \|f\|_{\mathbb{G}}$) and through scaling ($\lambda^{3-\frac{2}{p}} \|f(\lambda x)\|_{\mathbb{G}} = \|f\|_{\mathbb{G}}$). Then we have

$$\|e^{t\Delta} f\|_{L^\infty(dx)} \leq C \|f\|_{\mathbb{G}} t^{-\frac{3}{2} + \frac{1}{p}}$$

and this gives, for $1 \leq p \leq +\infty$, that $\mathbb{G}$ is embedded into the homogeneous Besov space $\dot{B}^{-3+\frac{2}{p}}_{\infty,\infty}$.

Here, we describe eternal solutions of a Stokes problem

$$\begin{cases} \partial_t \vec{u} = \Delta \vec{u} + \vec{f} - \vec{\nabla} p \\ \text{div } \vec{u} = 0 \end{cases} \tag{3}$$

first in a non-periodic setting (Proposition 1) then in a time-periodic setting (Proposition 3) :

Proposition 1
Let $\vec{f}$ be defined on $\mathbb{R}\times\mathbb{R}^3$ and belong to $L^1_t\dot{B}^{-1}_{\infty,\infty}$, or $L^\infty_t\dot{B}^{-3}_{\infty,\infty}$ or $L^{p,\infty}_t\dot{B}^{-3+\frac{2}{p}}_{\infty,\infty}$ ($1 < p < +\infty$). Then the Stokes problem (3) has a unique solution $\vec{u} \in L^\infty\dot{B}^{-1}_{\infty,\infty}$.

Before proving the proposition, we show how to get rid from the pressure through the Leray projection operator $\mathbb{P}$:

Lemma 1 *Let $\vec{f}$ and $\vec{u}$ be as in Proposition 1. Then $\vec{\nabla}p$ is given by*

$$\vec{\nabla}p = \vec{f} - \mathbb{P}\vec{f} = \vec{\nabla}\frac{1}{\Delta}\text{div } \vec{f}.$$

Proof :
First, we remark that $\frac{\partial_i\partial_j}{\Delta}$ is bounded on $\dot{B}^{-\sigma}_{\infty,\infty}$ for every $\sigma > 0$: if $f \in \dot{B}^{-\sigma}_{\infty,\infty}$, then

- $\partial_i\partial_j f \in \dot{B}^{-\sigma-2}_{\infty,\infty}$: $\|\partial_i\partial_j e^{t\Delta} f\|_\infty \leq Ct^{-1}\|e^{\frac{t}{2}\Delta} f\|_\infty \leq C' t^{-(1+\frac{\sigma}{2})}\|f\|_{\dot{B}^{-\sigma}_{\infty,\infty}}$
- $g = \int_0^1 \partial_i\partial_j e^{t\Delta} f\, dt + \int_1^{+\infty} \partial_i\partial_j e^{t\Delta} f\, dt$ is well defined and belongs to $\dot{B}^{-\sigma-2}_{\infty,\infty} + L^\infty$.
- g belongs to $\dot{B}^{-\sigma}_{\infty,\infty}$:

  $$\|e^{\tau\Delta} g\|_\infty \leq \int_0^\infty \|\partial_i\partial_j e^{(\tau+t)\Delta} f\|_\infty\, dt \leq C\|f\|_{\dot{B}^{-\sigma}_{\infty,\infty}} \int_0^\infty \frac{dt}{(t+\tau)^{1+\frac{\sigma}{2}}}$$

  so that $\|e^{\tau\Delta} g\|_\infty \leq C'\tau^{-\sigma/2}\|f\|_{\dot{B}^{-\sigma}_{\infty,\infty}}$.
- since $\lim_{t\to+\infty} \|e^{t\Delta} f\|_\infty = 0$, we find that $\int_0^{+\infty} \Delta e^{t\Delta} f\, dt = -f$, so that $-\Delta g = \partial_i\partial_j f$ and $g = -\frac{\partial_i\partial_j}{\Delta} f$.

Thus, we see that $\vec{g} = \vec{f} - \mathbb{P}\vec{f}$ is well defined and satisfies div $\vec{g} =$ div $\vec{f}$. On the other hand, taking the divergence of equation (3), we find that div $\vec{\nabla}p =$ div $\vec{f}$. Thus, $\vec{h} = \vec{\nabla}p - \vec{g}$ is both irrotational and divergence free, so that $\Delta\vec{h} = 0$. If we take $\phi \in \mathcal{D}(\mathbb{R}\times\mathbb{R}^3)$, we find that $\Delta(\phi * \vec{h}) = 0$ while $\phi * \vec{h}$ belongs to $L^\infty_t\dot{B}^{-1}_{\infty,\infty}$, so that $\phi * \vec{h}(t,x) = -\int_0^{+\infty} e^{\tau\Delta}\Delta(\phi * \vec{h})(t,.)\, d\tau = 0$. This gives $\phi * \vec{h} = 0$ for every ϕ, hence $\vec{h} = 0$. ◇

We may now prove Proposition 1 :

Proof :
If $\vec{u}$ exists, then the mapping $t \mapsto \vec{u}(t,.)$ should be continuous with values

in $\dot{B}^{-1}_{\infty,\infty} + \dot{B}^{-3}_{\infty,\infty}$ (as $\partial_t \vec{u}$ is locally integrable with values in $\dot{B}^{-1}_{\infty,\infty} + \dot{B}^{-3}_{\infty,\infty}$). Moreover, if $t_0 < t$, we have

$$\vec{u}(t,x) = e^{(t-t_0)\Delta}\vec{u}(t_0,.) + \int_{t_0}^{t} e^{(t-s)\Delta}\mathbb{P}\vec{f}(s,.)\,ds$$

with $\|e^{(t-t_0)\Delta}\vec{u}(t_0,.)\|_\infty \leq C(t-t_0)^{-1/2}\|\vec{u}\|_{L^\infty \dot{B}^{-1}_{\infty,\infty}}$. Letting t_0 go to $-\infty$, we find Kozono and Nakao's formula [10] :

$$\vec{u} = \int_{-\infty}^{t} e^{(t-s)\Delta}\mathbb{P}\vec{f}(s,.)\,ds \tag{4}$$

Now, let us define $\vec{u}$ as $\vec{u} = \int_{-\infty}^{t} e^{(t-s)\Delta}\mathbb{P}\vec{f}(s,.)\,ds$ and let us estimate $\vec{u}$ in $\dot{B}^{-1}_{\infty,\infty}$. For $1 \leq p \leq +\infty$, and $\tau > 0$, we have

$$\|e^{\tau\Delta}\vec{u}\|_\infty \leq C_p \int_{-\infty}^{t} \frac{1}{(\tau+t-s)^{\frac{3}{2}-\frac{1}{p}}}\|\vec{f}(s,.)\|_{\dot{B}^{-3+\frac{2}{p}}}\,ds.$$

Let $k_{p,\tau}(t) = 1_{t>0}\ (t+\tau)^{-\frac{3}{2}+\frac{1}{p}}$. Then $\|k_{p,1}\|_\infty = \tau^{-1/2}$ and $\|k_{p,\infty}\|_1 = 2\tau^{-1/2}$. If $1 < p < +\infty$, we remark that $(\frac{3}{2}-\frac{1}{p})\frac{p}{p-1} > 1$, so that $k_{p,\tau} \in L^{\frac{p}{p-1},1}$ and $\|k_{p,\tau}\|_{L^{\frac{p}{p-1},1}} = C_p\tau^{-1/2}$. Since convolution mapts $L^\infty \times L^1$, $L^1 \times L^\infty$ and $L^{\frac{p}{p-1},1} \times L^{p,\infty}$ to L^∞, we find that $\vec{u}$ belongs to $L^\infty \dot{B}^{-1}_{\infty,\infty}$. $\diamond$

A special case is the case of constant forces :

Proposition 2
If $\vec{u}$ belongs to $L^\infty \dot{B}^{-1}_{\infty,\infty}$ and is a solution of the Stokes problem (3) with a constant force $\vec{f}(x)$, then $\vec{u}$ is constant and $\mathbb{P}\vec{f}$ belongs to $\dot{B}^{-3}_{\infty,\infty}$.

Proof :
If $\vec{f}$ is constant, formula (4) gives

$$\vec{u} = \int_0^{+\infty} e^{s\Delta}\mathbb{P}\vec{f}\,ds$$

so that $\vec{u}$ is constant. Moreover $\Delta\vec{u} = -\mathbb{P}\vec{f}$, so that $\mathbb{P}\vec{f} \in \dot{B}^{-3}_{\infty,\infty}$. $\diamond$

When we consider a time-periodic force $\vec{f}_{\rm per}$ (with period T), we then split the force between its (constant) mean value $\vec{f}_0(x) = \frac{1}{T}\int_0^T \vec{f}_{\rm per}(t,x)\,dt$ and its fluctuation $\vec{f}_1 = \vec{f}_{\rm per} - \vec{f}_0$. The case of $\vec{f}_0$ has been settled by Proposition 2; thus we consider only the case when $\vec{f}_{per} = \vec{f}_1$:

Proposition 3

Let $\vec{f}_{\rm per}$ be a time-periodic force with period T and with mean value 0 (i.e. $\frac{1}{T}\int_0^T \vec{f}_{\rm per}(t,x)\,dt = 0$) and belong to $L^1_{\rm per}\dot{B}^{-1}_{\infty,\infty}$, or $L^\infty_t\dot{B}^{-3}_{\infty,\infty}$ or $L^{p,\infty}_{\rm per}\dot{B}^{-3+\frac{2}{p}}_{\infty,\infty}$ ($1 < p < +\infty$). Then the Stokes problem (3) has a unique (periodic) solution $\vec{u}_{\rm per} \in L^\infty\dot{B}^{-1}_{\infty,\infty}$.

Proof :

We shall prove that $\vec{u}$ may still be defined through formula (4). If the formula defines a distribution $\vec{u}$, this one will be time-periodic. Thus, we may assume that $t \in (0,T)$ and try and prove that the formula is well defined and that its results belongs to $L^\infty_t\dot{B}^{-1}_{\infty,\infty}$.

We then split $\vec{f}_{\rm per}$ into three pieces : $\vec{f}_1 = 1_{-T<t<T}\vec{f}_{\rm per}$, $\vec{f}_2 = 1_{t<-T}\vec{f}_{\rm per}$ and $\vec{f}_3 = 1_{t>T}\vec{f}_{\rm per}$. $\vec{f}_3$ does not contribute to the values of $\vec{u}$ on $(0,T)$ We thus find that $\vec{u}(t,x) = \vec{u}_1(t,x) + \vec{u}_2(t,x)$ with

$$\vec{u}_1(t,x) = \int_{-\infty}^t e^{(t-s)\Delta}\mathbb{P}\vec{f}_1(s,.)\,ds$$

and

$$\vec{u}_2(t,x) = \int_{-\infty}^{-T} e^{(t-s)\Delta}\mathbb{P}\vec{f}_{\rm per}(s,.)\,ds.$$

By Proposition 1, we already know that $\vec{u}_1$ belongs to $L^\infty_t\dot{B}^{-1}_{\infty,\infty}$. Moreover, if $\vec{F} = \int_0^t \mathbb{P}\vec{f}_{\rm per}(s,.)\,ds$, we have that $\vec{F}$ is still T-periodic (as the mean value of $\vec{f}_{\rm per}$ is assumed to be 0). We then find that

$$\vec{u}_2(t,x) = \sum_{k=1}^{+\infty}\int_{-(k+1)T}^{-kT} \Delta e^{(t-s)\Delta}\vec{F}(s,.)\,ds$$

In the case $1 < p$, we find that $\vec{F}$ belongs to $L^\infty\dot{B}^{-3+\frac{2}{p}}_{\infty,\infty}$ and that

$$\|\vec{u}_2(t,.)\|_{\dot{B}^{-1}_{\infty,\infty}} \le C_p\|\vec{F}\|_{L^\infty_t\dot{B}^{-3+\frac{2}{p}}_{\infty,\infty}}\int_T^{+\infty}\frac{1}{s^{2-\frac{1}{p}}}\,ds$$

In the case $p = 1$, we call $\vec{F}_0$ the mean value of $\vec{F}$ and we define $\vec{G}(t,.) = \int_0^t(\vec{F}(s,.) - \vec{F}_0)\,ds$. We thus find that

$$\vec{u}_2(t,x) = -e^{(t+T)\Delta}\vec{F}_0 + \sum_{k=1}^{+\infty}\int_{-(k+1)T}^{-kT}\Delta^2 e^{(t-s)\Delta}\vec{G}(s,.)\,ds$$

and

$$\|\vec{u}_2(t,.)\|_{\dot{B}^{-1}_{\infty,\infty}} \le C(\|\vec{F}_0\|_{\dot{B}^{-1}_{\infty,\infty}} + \|\vec{G}\|_{L^\infty_t\dot{B}^{-1}_{\infty,\infty}}\int_T^{+\infty}\frac{1}{s^2}\,ds).$$

◇

2 From Stokes to Navier–Stokes.

As usual since the work of Oseen [16], the Navier-Stokes problem

$$\begin{cases} \partial_t \vec{u} + \text{ div } (\vec{u} \otimes \vec{u}) = \Delta \vec{u} + \vec{f} - \vec{\nabla} p \\ \text{div } \vec{u} = 0 \end{cases} \tag{5}$$

is seen as a Stokes problem

$$\begin{cases} \partial_t \vec{u} = \Delta \vec{u} + \vec{f} - \vec{g} - \vec{\nabla} p \\ \text{div } \vec{u} = 0 \\ \vec{g} = \text{div } (\vec{u} \otimes \vec{u}) \end{cases} \tag{6}$$

and may be solved by a Picard iterative scheme : $\vec{u}$ will be the limit of the solutions $\vec{U}_n$ of the Stokes problem

$$\begin{cases} \partial_t \vec{U}_n = \Delta \vec{U}_n + \vec{F}_n - \vec{\nabla} P_n \\ \text{div } \vec{U}_n = 0 \end{cases}$$

with $\vec{F}_0 = \vec{f}$ and $\vec{F}_{n+1} = \vec{f} - \text{div } (\vec{U}_n \otimes \vec{U}_n)$.

Using formula (4), we thus study the sequence defined by

$$\vec{U}_0 = \int_{-\infty}^{t} e^{(t-s)\Delta} \mathbb{P} \vec{f}(s,.)\, ds$$

and

$$\vec{U}_{n+1} = \vec{U}_0 - B(\vec{U}_n, \vec{U}_n)$$

with

$$B(\vec{U}, \vec{V}) = \int_{-\infty}^{t} e^{(t-s)\Delta} \mathbb{P} \text{div } (\vec{U} \otimes \vec{V})\, ds$$

Of course, we may no longer work in the space $L^\infty_t \dot{B}^{-1}_{\infty,\infty}$, as we need to define the pointwise product $\vec{U} \otimes \vec{V}$ for $\vec{U}$ and $\vec{V}$ in the space where we look for solutions. If we still want to keep the invariances of the Navier–Stokes equations for a space of solutions $\mathbb{W}$ for $\vec{u}$, we may look for spaces $\mathbb{W} = L^p_t \mathbb{Z}_x$ (or $L^{p,\infty}_t \mathbb{Z}_x$, when $2 < p < \infty$) where the norm of $\mathbb{Z}$ is invariant through space translations ($\|u(x - x_0)\|_{\mathbb{Z}} = \|u\|_{\mathbb{Z}}$) and through scaling ($\lambda^{1-\frac{2}{p}} \|u(\lambda x)\|_{\mathbb{Z}} = \|u\|_{\mathbb{Z}}$), and moreover $\mathbb{Z}$ is continuously embedded into L^2_{loc} (so that the pointwise produc is well defined). We then have that, for $u \in \mathbb{Z}$ and for any ball $B(x_0, R)$,

$$\int_{B(x_0,R)} |u(y)|^2\, dy = R^3 \int_{B(0,1)} \left| u(R(y + \frac{x_0}{R})) \right|^2\, dy \le R^{1+\frac{4}{p}} \|u\|^2_{\mathbb{Z}}$$

Thus, we get that $\mathbb{Z}$ is embedded into the homogeneous Morrey space $\dot M^{2,3\frac{p}{p-2}}$.

As the bilinear operator B is not bouded on $L^\infty_t \dot M^{2,3}$ nor on $L^{p,\infty}_t \dot M^{2,3\frac{p}{p-2}}$ (nor on $L^{p,\infty}_{\rm per} \dot M^{2,3\frac{p}{p-2}}$), we shall work on smaller subspaces. We define $\dot M^{L^{q,\infty},r}(\mathbb{R}^3)$ as the homogeneous Morry space (for $1 < q \le r < +\infty$) of the locally integrable functions such that

$$\sup_{x_0\in\mathbb{R}^3, R>0} R^{3(\frac{1}{q}-\frac{1}{r})} \|1_{B(x_0,R)} f\|_{L^{q,\infty}} < +\infty.$$

When $q = r$, we get the Lorentz space $L^{r,\infty}$. We have the following inclusions, for $2 < p \le \infty$ and $2 < q < 3\frac{p}{p-2}$:

$$L^{3\frac{p}{p-2}} \subset L^{3\frac{p}{p-2},\infty} \subset \dot M^{L^{q,\infty},3\frac{p}{p-2}} \subset \dot M^{2,3\frac{p}{p-2}}.$$

We shall use as well homogeneous Sobolev spaces $\dot H^s$ with $1/2 \le s < 3/2$ and the spaces $\mathcal{V}^{2,\sigma}$ of pointwise multipliers from $\dot H^\sigma$ to L^2 [13] for $0 < \sigma \le 1$. We have the Sobolev embedding

$$\dot H^s \subset L^{3\frac{p}{p-2}} \quad \text{for } s = \frac{1}{2} + \frac{2}{p}$$

and the Fefferman–Phong inequality

$$\dot M^{L^{q,\infty},3\frac{p}{p-2}} \subset \mathcal{V}^{2,\sigma} \subset \dot M^{2,3\frac{p}{p-2}} \quad \text{for } 2 < q < 3\frac{p}{p-2} \text{ and } \sigma = \frac{p-2}{p}.$$

3 Maximal functions and the Navier–Stokes problem.

In this section, we consider two Banach spaces of distributions on $\mathbb{R}^3$, $\mathcal{X} \subset L^2_{\rm loc}$ and $\mathcal{Y}$, such that :

- (A1) The pointwise product is bounded from $\mathcal{X} \times \mathcal{X}$ to $\mathcal{Y}$ and from $L^\infty \times \mathcal{X}$ to $\mathcal{X}$.
- (A2) The Riesz transforms are bounded on $\mathcal{X}$.
- (A3) The Hardy–Littlewood maximal function is bounded on $\mathcal{X}$.
- (A4) The operator $\frac{1}{\sqrt{-\Delta}}$ is bounded from $\mathcal{Y}$ to $\mathcal{X}$.

A consequence of assumptions (A1) and (A4) is the embedding $\mathcal{X} \subset \mathcal{V}^{2,1}$.

Proposition 4
Let $\mathcal{X}$ and $\mathcal{Y}$ satisfy assumptions (A1) and (A4). Let $\mathcal{E}$ be the space of divergence free vector fields $\vec{u}$ such that $\sup_{t\in\mathbb{R}} |\vec{u}(t,x)| \in \mathcal{X}$. Then the bilinear operator B is bounded on $\mathcal{E} \times \mathcal{E}$.

Proof :
The proof follows the proof in the case of $\mathcal{X} = L^3$ given by Calderón [1] and Cannone [2]. We have, writing $U_{\max}(x) = \sup_{t\in\mathbb{R}} |\vec{U}(t,x)|$ and $V_{\max}(x) = \sup_{t\in\mathbb{R}} |\vec{V}(t,x)|$,

$$\begin{aligned} |B(\vec{U},\vec{V})(t,x)| &\leq C \int_{-\infty}^{t} \int_{\mathbb{R}^3} \frac{1}{(t-s)^2 + |x-y|^4} |\vec{U}(t,y)|\, |\vec{V}(t,y)|\, dy\, ds \\ &\leq C \int_{\mathbb{R}^3} \left(\int_{-\infty}^{t} \frac{1}{(t-s)^2 + |x-y|^4}\, ds \right) U_{\max}(y)\, V_{\max}(y)\, dy \\ &= \frac{\pi}{2} C \int \frac{1}{|x-y|^2} U_{\max}(y)\, V_{\max}(y)\, dy \end{aligned}$$

and thus

$$\sup_{t\in\mathbb{R}} |B(\vec{U},\vec{V})(t,x)| \leq C \frac{1}{\sqrt{-\Delta}} (U_{\max}\, V_{\max})(x)$$

and

$$\| \sup_{t\in\mathbb{R}} |B(\vec{U},\vec{V})(t,x)| \|_{\mathcal{X}} \leq C \|U_{\max}\, V_{\max}\|_{\mathcal{Y}} \leq C' \|U_{\max}\|_{\mathcal{X}} \|V_{\max}\|_{\mathcal{X}}$$

$\diamond$

Proposition 5
Let $\mathcal{X}$ and $\mathcal{Y}$ satisfy assumptions (A1) to (A4). Let $\mathcal{E}$ be the space of divergence free vector fields $\vec{u}$ such that $\sup_{t\in\mathbb{R}} |\vec{u}(t,x)| \in \mathcal{X}$. Then there exists a positive constant $\epsilon_{\mathcal{X}}$ such that : if $\vec{f}$ is a time-dependent vector field on $\mathbb{R} \times \mathbb{R}^3$ which belongs to $L^1_t \mathcal{X}$ with

$$\int_{\mathbb{R}} \|\vec{f}\|_{\mathcal{X}}\, dt < \epsilon_{\mathcal{X}}$$

then there exists a solution $\vec{u}$ of the Navier-Stokes problem (5) such that $\vec{u} \in \mathcal{E}$.

Proof :
$\vec{u}$ will be the limit of the sequence $\vec{U}_n$ defined by

$$\vec{U}_0 = \int_{-\infty}^{t} e^{(t-s)\Delta} \mathbb{P} \vec{f}(s,.)\, ds$$

and

$$\vec{U}_{n+1} = \vec{U}_0 - B(\vec{U}_n, \vec{U}_n).$$

This sequence will be convergent as soon as $\vec{U}_0 \in \mathcal{E}$ and $\|\vec{U}_0\|_{\mathcal{E}} \leq \frac{1}{4C_0}$ where C_0 is the norm of the bilinear operator B on $\mathcal{E}$. On the other hand, we have

$$|\vec{U}_0(t,x)| \leq \int_{-\infty}^{t} M_{\mathbb{P}\vec{f}(s,.)}(x)\, ds$$

where M_f is the Hardy–Littlewood maximal function of f. Thus we find

$$\begin{aligned} \|\sup_{t\in\mathbb{R}} |\vec{U}_0(t,x)|\|_{\mathcal{X}} &\leq C\|\int_{\mathbb{R}} M_{\mathbb{P}\vec{f}(s,.)}(x)\, ds\|_{\mathcal{X}} \\ &\leq C\int_{\mathbb{R}} \|M_{\mathbb{P}\vec{f}(s,.)}\|_{\mathcal{X}}\, ds \leq C' \int_{\mathbb{R}} \|\vec{f}(s,.)\|_{\mathcal{X}}\, ds \end{aligned}$$

and the proposition is proved. $\diamond$

Theorem 1
Let $\mathcal{X}$ and $\mathcal{Y}$ satisfy assumptions (A1) to (A4). Let $\mathcal{E}$ be the space of divergence free vector fields $\vec{u}$ such that $\sup_{t\in\mathbb{R}} |\vec{u}(t,x)| \in \mathcal{X}$. Then there exists a positive constant $\epsilon_{\mathcal{X},T}$ such that : if $\vec{f}_{\rm per}$ is a time-periodic vector field on $\mathbb{R}\times\mathbb{R}^3$ such that

- *the mean value $\vec{f}_0 = \frac{1}{T}\int_0^T \vec{f}_{\rm per}(s,.)\, ds$ belongs to $\dot{B}^{-3}_{\infty,\infty}$ and satisfies $\frac{1}{\Delta}\vec{f}_0 \in \mathcal{X}$ with*

$$\|\frac{1}{\Delta}\mathbb{P}\vec{f}_0\|_{\mathcal{X}} < \epsilon_{\mathcal{X},T}$$

- *the fluctuation $\vec{f}_1 = \vec{f}_{\rm per} - \vec{f}_0$ belongs to $L^1_{\rm per}\mathcal{X}$ with*

$$\int_0^T \|\mathbb{P}\vec{f}_1\|_{\mathcal{X}}\, dt < \epsilon_{\mathcal{X},T}$$

then there exists a time-periodic solution $\vec{u}_{\rm per}$ of the Navier-Stokes problem (5) such that $\vec{u}_{\rm per} \in \mathcal{E}$.

Proof :
Again, we must check that $\vec{U}_0 = \int_{-\infty}^t e^{(t-s)\Delta}\mathbb{P}\vec{f}(s,.)\, ds$ belongs to $\mathcal{E}$. The proof follows the lines of the proof of Proposition 3. We have $\vec{U}_0 = \frac{1}{\Delta}\mathbb{P}\vec{f}_0 + \vec{V}_0$ with $\vec{V}_0 = \int_{-\infty}^t e^{(t-s)\Delta}\mathbb{P}\vec{f}_1(s,.)\, ds$.

We study $\vec{V}_0$ on the period interval $t \in [0,T]$. We then split $\vec{V}_0(t,x) = \vec{V}_1(t,x) + \vec{V}_2(t,x)$ with

$$\vec{V}_1(t,x) = \int_{-\infty}^{t} e^{(t-s)\Delta} 1_{[-T,T]}(s) \mathbb{P} \vec{f}_1(s,.)\, ds$$

and

$$\vec{V}_2(t,x) = \int_{-\infty}^{-T} e^{(t-s)\Delta} \mathbb{P} \vec{f}_1(s,.)\, ds.$$

By the proof of Proposition 5, we already know that $\vec{V}_1$ belongs to $\mathcal{E}$. Moreover, if $\vec{F} = \int_0^t \mathbb{P}\vec{f}_1(s,.)\, ds$, we have that $\vec{F}$ is still T-periodic (as the mean value of $\vec{f}_1$ is equal to 0). Moreover, we call $\vec{F}_0$ the mean value of $\vec{F}$ and we define $\vec{G}(t,.) = \int_0^t (\vec{F}(s,.) - \vec{F}_0)\, ds$. We have obviously $\vec{F}_0 \in \mathcal{X}$ and $\vec{G} \in L^1_{\text{per}}\mathcal{X}$. We thus find that

$$\vec{V}_2(t,x) = -e^{(t+T)\Delta}\vec{F}_0 + \sum_{k=1}^{+\infty} \int_{-(k+1)T}^{-kT} \Delta^2 e^{(t-s)\Delta} \vec{G}(s,.)\, ds$$

satisfies on $[0,T]$

$$|\vec{V}_2(t,x)| \le C(M_{\vec{F}_0}(x) + \int_0^T M_{\vec{G}(s,.)}(x)\, ds \sum_{k=1}^{+\infty} \frac{1}{(kT)^2}).$$

The theorem is proved. $\diamond$

Examples and comments :
We shall give some examples of spaces $\mathcal{X}$ and $\mathcal{Y}$ which satisfy (A1) to (A4) and such that $L^3 \subset \mathcal{X}$.

- First, let us remark that if $\vec{f}$ is divergence-free and belongs to $L^1_{\text{per}}\mathcal{X}$, and if moreover the support of $\vec{f}$ is bounded : $\vec{f}(t,x) = 0$ if $|x| \ge R$, then its mean value belongs to $\dot{B}^{-3}_{\infty,\infty}$ and satisfies $\frac{1}{\Delta}\vec{f}_0 \in \mathcal{X}$. Indeed, $\vec{f}_0 \in \mathcal{X} \subset \mathcal{V}^{2,1} \subset \dot{M}^{2,3}$ and the support of $\vec{f}_0$ is contained in the ball $\bar{B}(0,R)$. Thus, $\vec{f}_0 \in L^1$. Since $\vec{f}_0$ is divergence-free, we find that $\int \vec{f}_0\, dx = 0$. Moreover, $\vec{f}_0 \in L^2$ and has compact support. Hence, $\vec{f}_0$ belongs to the Hardy space $\mathcal{H}^1$, hence $\frac{1}{\Delta}\vec{f}_0 \in L^3 \subset \mathcal{X}$.
- Calderón and Cannone's theorems [1, 2] lead to the case $\mathcal{X} = L^3$ and $\mathcal{Y} = L^{3/2}$.
- Yamazaki's theorem [19] would correspond to the case $\mathcal{X} = L^{3,\infty}$ and $Y = L^{3/2,\infty}$.

- The results of Phuc and Phan on steady solutions [17] would correspond to $\mathcal{X} = \mathcal{V}^{2,1} = \mathcal{M}(\dot H^1 \mapsto L^2)$ and $\mathcal{Y} = \mathcal{M}(\dot H^1 \mapsto \dot H^{-1})$.
- One could consider as well Morrey spaces $\mathcal{X} = \dot M^{p,3}$ and $\mathcal{Y} = \dot M^{p/2,3/2}$ $(2 < p \leq 3)$ or Morrey–Lorentz spaces $\mathcal{X} = \dot M^{L^{p,\infty},3}$ and $\mathcal{Y} = \dot M^{L^{p/2,\infty},3/2}$ $(2 < p \leq 3)$.

One last remark is that, in most cases, our solutions are regular enough to justify the equality $\operatorname{div} (\vec u \otimes \vec u) = (\vec u.\vec\nabla)\vec u$:

Theorem 2
Let $\mathcal{X}$ and $\mathcal{Y}$ satisfy assumptions (A1) to (A4). Let $\mathcal{E}$ be the space of divergence free vector fields $\vec u$ such that $\sup_{t\in\mathbb{R}} |\vec u(t,x)| \in \mathcal{X}$. Then there exists a positive constant $\eta_{\mathcal{X},T}$ such that : if $\vec f_{\rm per}$ is a time-periodic vector field on $\mathbb{R}\times\mathbb{R}^3$ such that

- *the mean value $\vec f_0 = \frac{1}{T}\int_0^T \vec f_{\rm per}(s,.)\,ds$ belongs to $\dot B^{-3}_{\infty,\infty}$ and satisfies $\frac{1}{\Delta}\vec f_0 \in \mathcal{X}$ with*
$$\|\frac{1}{\Delta}\mathbb{P}\vec f_0\|_{\mathcal{X}} < \eta_{\mathcal{X},T}$$
- *$\vec f_{\rm per}$ belongs to $L^1_{\rm per}\mathcal{X}$ with*
$$\int_0^T \|\mathbb{P}\vec f_{\rm per}\|_{\mathcal{X}}\,dt < \eta_{\mathcal{X},T}$$

then there exists a time-periodic solution $\vec u_{\rm per}$ of the Navier-Stokes problem (5) such that $\vec u_{\rm per} \in \mathcal{E}$ and $\vec\nabla \otimes \vec u_{\rm per} \in L^{2,\infty}_{\rm per}\mathcal{X}$.

This solution satisfies $\operatorname{div} (\vec u_{\rm per} \otimes \vec u_{\rm per}) = (\vec u_{\rm per}.\vec\nabla)\vec u_{\rm per}$

Proof :
First we check that $\partial_j \vec U_0$ belongs to $L^{2,\infty}_{\rm per}\mathcal{X}$ for $j = 1,\ldots,3$. We write again $\vec f_{\rm per} = \vec f_0 + \vec f_1$ and $\vec U_0 = \frac{1}{\Delta}\mathbb{P}\vec f_0 + \vec V_0$. Let us remark that $\vec f_0 \in \mathcal{X}$. We have
$$\partial_j(\frac{1}{\Delta}\mathbb{P}\vec f_0) = \int_0^{+\infty} \partial_j e^{t\Delta}\mathbb{P}\vec f_0\,dt$$
so that
$$|\partial_j(\frac{1}{\Delta}\mathbb{P}\vec f_0)(x)| \leq \int_0^1 C\frac{1}{t^{1/2}} M_{\mathbb{P}\vec f_0}(x)\,dt + \int_1^{+\infty} C\frac{1}{t^{3/2}} M_{\frac{1}{\Delta}\mathbb{P}\vec f_0}(x)\,dt$$

and thus $\partial_j(\frac{1}{\Delta}\mathbb{P}\vec{f}_0) \in \mathcal{X}$. We split again $\vec{V}_0$ into $\vec{V}_1 + \vec{V}_2$ on $[0, T]$. We have

$$|\partial_j \vec{V}_1(t,x)| \leq C \int_{-\infty}^{t} \frac{1}{(t-s)^{1/2}} 1_{[-T,T]}(s) M_{\mathbb{P}\vec{f}_1(s,.)}(x)\, ds$$

and thus

$$\|\partial_j \vec{V}_1(t,.)\|_{\mathcal{X}} \leq C \int_{-\infty}^{t} \frac{1}{(t-s)^{1/2}} 1_{[-T,T]}(s) \|\mathbb{P}\vec{f}_1(s,.)\|_{\mathcal{X}}\, ds$$

The function $1_{t>0} t^{-1/2}$ belongs to $L^{2,\infty}$ and the function $1_{[-T,T]}(t)\|\mathbb{P}\vec{f}_1(t,.)\|_{\mathcal{X}}$ belongs to L^1, hence their convoution still belongs to $L^{2,\infty}$. The case of $\partial_j \vec{V}_2$ is still simpler, as we have on $[0, T]$,

$$|\partial_j \vec{V}_2(t,x)| \leq C(\frac{1}{T^{1/2}} M_{\vec{F}_0}(x) + \int_0^T M_{\vec{G}(s,.)}(x)\, ds \sum_{k=1}^{+\infty} \frac{1}{(kT)^{5/2}}).$$

Thus $\partial_j \vec{U}_0 \in L^{2,\infty}_{\rm per}\mathcal{X}$.

The next step is to prove that the bilinear operator B is bounded on the space $\mathcal{F} = \{\vec{U} \in \mathcal{E} \ / \ \vec{\nabla} \otimes \vec{U} \in L^{2,\infty}_{\rm per}\mathcal{X}$ and $\operatorname{div} \vec{U} = 0\}$. First, we notice that for any $\vec{U} \in \mathcal{F}$, $\varphi \in \mathcal{D}(\mathbb{R}^3)$ and $1 < r < 2$ we have $\varphi \partial_j \vec{U} \in L^r_{\rm per} L^2$ while $\varphi \vec{U}$ belongs to $L^{\frac{r}{r-1}}_{\rm per} L^2$. This is enough to get that, when $\vec{U}$ and $\vec{V}$ belong to $\mathcal{F}$, we have $\partial_k(\vec{U} \otimes \vec{V}) = (\partial_k \vec{U}) \otimes \vec{V} + \vec{U} \otimes (\partial_k \vec{V})$ for $k = 1, \ldots, 3$. We have moreover :

$$\|U\|_{[\mathcal{X}, L^\infty]_{1/2,\infty}} \leq C \|U\|_{\mathcal{X}}^{1/2} \|\vec{\nabla} U\|_{\mathcal{X}}^{1/2} \tag{7}$$

Indeed, we write $U = -\int_0^{+\infty} \Delta e^{s\Delta} U\, ds$ and then, for $A > 0$,

$$U = -\sum_{k=1}^{3} \int_0^A \partial_k e^{s\Delta}(\partial_k U)\, ds - \int_A^{+\infty} \Delta e^{s\Delta} U\, ds$$

with

$$\|\sum_{k=1}^{3} \int_0^A \partial_k e^{s\Delta}(\partial_k U)\, ds\|_{\mathcal{X}} \leq C\sqrt{A} \|\vec{\nabla} U\|_{\mathcal{X}}$$

and (since $\mathcal{X} \subset \mathcal{V}^{1,2} \subset \dot{B}^{-1}_{\infty,\infty}$)

$$\|\int_A^{+\infty} \Delta e^{s\Delta} U\, ds\|_\infty \leq C \frac{1}{\sqrt{A}} \|U\|_{\mathcal{X}}.$$

Thus (7) is proved

We thus get that, for $\vec U$ and $\vec V$ in $\mathcal{F}$, div $(\vec U \otimes \vec V) = (\vec U.\vec\nabla)\vec V \in L^{4/3,\infty}_{\rm per}[\mathcal{Y},\mathcal{X}]_{1/2,\infty}$ (since the pointwise product maps $\mathcal{X}\times\mathcal{X}$ to $\mathcal{Y}$ and $\mathcal{X}\times L^\infty$ to $\mathcal{X}$ hence $\mathcal{X}\times[\mathcal{X},L^\infty]_{1/2,\infty}$ to $[\mathcal{Y},\mathcal{X}]_{1/2,\infty}$).

We want to estimate $\|\partial_j B(\vec U,\vec V)(t,.)\|_{\mathcal{X}} \le C\|\int_{-\infty}^t \partial_j e^{(t-s)\Delta}(\vec U.\vec\nabla\vec V)\,ds\|_{\mathcal{X}}$ for $t\in[0,T]$. We write $\vec U.\vec\nabla\vec V = \vec W = \vec W_0 + \vec W_1$ where $\vec W_0$ is the mean value of $\vec W$ and $\vec W_1$ its fluctuation. We have $\vec W \in L^{4/3,\infty}_{\rm per}[\mathcal{Y},\mathcal{X}]_{1/2,\infty}\cap L^{2,\infty}_{\rm per}\mathcal{Y}$. Then $\vec W_0\in[\mathcal{Y},\mathcal{X}]_{1/2,\infty}\cap\mathcal{Y}$ and $\vec W_1\in L^{4/3,\infty}_{\rm per}[\mathcal{Y},\mathcal{X}]_{1/2,\infty}$. We have

$$\|\int_{-\infty}^t \partial_j e^{(t-s)\Delta}\vec W_0\,ds\|_{\mathcal{X}} = \|\frac{\partial_j}{\sqrt{-\Delta}}\frac{1}{\sqrt{-\Delta}}\vec W_0\|_{\mathcal{X}} \le C\|\vec W_0\|_{\mathcal{Y}}.$$

Let $\vec W_2 = \int_0^t \vec W_1(s,.)\,ds$. Then $\vec W_2\in L^\infty_t[\mathcal{Y},\mathcal{X}]_{1/2,\infty}$. We have, for $0\le t\le T$,

$$\int_{-\infty}^t \partial_j e^{(t-s)\Delta}\vec W_1\,ds = \int_{-\infty}^t \partial_j e^{(t-s)\Delta}1_{[-T,T]}(s)\vec W_1\,ds + \sum_{k=1}^{+\infty}\int_{-(k+1)T}^{-kT}\Delta\partial_j e^{(t-s)\Delta}\vec W_2\,ds.$$

$e^{\tau\Delta}$ maps $\mathcal{Y}$ to $\mathcal{X}$ with norm $O(\frac{1}{\sqrt\tau})$ and $\mathcal{X}$ to $\mathcal{X}$ with norm $O(1)$, hence $[\mathcal{Y},\mathcal{X}]_{1/2,\infty}$ to $\mathcal{X}$ with norm $O(\frac{1}{\tau^{1/4}})$. Thus, we find that

$$\begin{aligned}\|\int_{-\infty}^t \partial_j e^{(t-s)\Delta}\vec W_1\,ds\|_{\mathcal{X}} \le & C\int_{-\infty}^t \frac{1}{(t-s)^{3/4}}1_{[-T,T]}(s)\|\vec W_1(s,.)\|_{[\mathcal{Y},\mathcal{X}]_{1/2,\infty}}\,ds \\ & + C\|\vec W_2\|_{L^\infty_t[\mathcal{Y},\mathcal{X}]_{1/2,\infty}}\sum_{k=1}^{+\infty}\frac{T}{(kT)^{5/4}}\end{aligned}$$

As the convolution maps $L^{4/3,\infty}\times L^{4/3,\infty}$ to $L^{2,\infty}$, we find that $\partial_j B(\vec U,\vec V)\in L^{2,\infty}$. Thus, B is bounded on $\mathcal{F}$ and the theorem is proved. $\diamond$

4 Real interpolation theory and the Navier–Stokes problem.

Theorem 1 is based on Proposition 4. We have shown that B is bounded on $\mathcal{E}$. We have the enbedding $\mathcal{E}\subset L^\infty_t\mathcal{X}$. One may ask the question whether B is bounded on the larger space $L^\infty_t\mathcal{X}$. The answer is negative, however, as it has been proved for $\mathcal{X}=L^3$ [15].

The answer is positive if we add further assumptions to (A1)–(A4) :

- (A5) the convolution is bounded from $L^1\times\mathcal{Y}$ to $\mathcal{Y}$
- (A6) $\mathcal{X}$ and $\mathcal{Y}$ satisfy $[\mathcal{Y},L^\infty]_{1/2,\infty}\subset\mathcal{X}$.

Proposition 6
Let $\mathcal{X}$ and $\mathcal{Y}$ satisfy assumptions (A1) to (A6). Then the bilinear operator B is bounded on $L^\infty_t\mathcal{X}$.

Proof :
The proof follows the proof in the case of $\mathcal{X} = L^{3,\infty}$ given by Meyer [14] and Yamazaki [19]. We write, for $A > 0$,

$$\begin{aligned}\| \int_{-\infty}^{t-A} e^{(t-s)\Delta}\mathbb{P}\,\mathrm{div}\,(\vec{U}\otimes\vec{V})\,ds\|_\infty &\leq C\int_{-\infty}^{t-A}\frac{1}{(t-s)^{3/2}}\|\vec{U}(s,.)\|_{\mathcal{X}}\|\vec{V}(s,.)\|_{\mathcal{X}}\,ds\\ &\leq 2C\frac{1}{\sqrt{A}}\|\vec{U}\|_{L^\infty_t\mathcal{X}}\|\vec{V}\|_{L^\infty_t\mathcal{X}}\end{aligned}$$

and

$$\begin{aligned}\| \int_{t-A}^{t} e^{(t-s)\Delta}\mathbb{P}\,\mathrm{div}\,(\vec{U}\otimes\vec{V})\,ds\|_{\mathcal{Y}} &\leq C\int_{t-A}^{t}\frac{1}{(t-s)^{1/2}}\|\vec{U}(s,.)\|_{\mathcal{X}}\|\vec{V}(s,.)\|_{\mathcal{X}}\,ds\\ &\leq 2C\sqrt{A}\|\vec{U}\|_{L^\infty_t\mathcal{X}}\|\vec{V}\|_{L^\infty_t\mathcal{X}}\end{aligned}$$

We thus get

$$\|B(\vec{U},\vec{V})(t,.)\|_{[\mathcal{Y},L^\infty]_{1/2,\infty}} \leq C\|\vec{U}\|_{L^\infty_t\mathcal{X}}\|\vec{V}\|_{L^\infty_t\mathcal{X}}.$$

◇

Propositions 1 and 5 may be further extended to the following case :

Proposition 7
Let $\mathcal{X}$ satisfy (A2). Let $\vec{f}$ be defined on $\mathbb{R}\times\mathbb{R}^3$ and belong to $L^1_t\mathcal{Z}_1$, or $L^\infty_t\mathcal{Z}_\infty$ or $L^{p,\infty}_t\mathcal{Z}_p$ $(1 < p < +\infty)$. Assume that $\mathcal{Z}_p$ satisfies the following assumptions :

- *(A7) the convolution is bounded from $L^1\times\mathcal{Z}_p$ to $\mathcal{Z}_p$*
- *(A8) $\mathcal{Z}_p \subset \dot{B}^{-3+\frac{2}{p}}_{\infty,\infty}$*
- *(A9) $\mathcal{Z}_1 \subset \mathcal{X}$, or, for $1 < p \leq \infty$, $\mathcal{Z}_p = (-\Delta)^{\alpha/2}\mathcal{Z}_{p,\alpha}$ with $0 \leq \alpha < 2-\frac{2}{p}$ and $[\mathcal{Z}_{p,\alpha}, L^\infty]_{\frac{p(2-\alpha)-2}{p(3-\alpha)-2},\infty} \subset \mathcal{X}$.*

Then the Stokes problem (3) has a unique solution $\vec{u}\in L^\infty_t\mathcal{X}$.

Proof :
We already know the case $p = 1$. So, we assume $p > 1$. By Proposition 1, we already know that $\vec{u} = \mathbb{P}\vec{v}$ with $\vec{v} = \int_{-\infty}^{t} e^{(t-s)\Delta}\vec{f}(s,.)\,ds$. To estimate the norm of $\vec{v}(t,.)$ in $\mathcal{X}$, we split the integral between $s < t - A$ and $s > t - A$ (where $A > 0$), we get

$$\begin{aligned}\|\int_{-\infty}^{t-A} e^{(t-s)\Delta}\vec{f}(s,.)\,ds\|_\infty &\leq C\int_{-\infty}^{t-A}\frac{1}{(t-s)^{3/2-1/p}}\|\vec{f}(s,.)\|_{\mathcal{Z}_p}\,ds\\ &\leq C'\|1_{t>0}\frac{1}{(t+A)^{3/2-1/p}}\|_{L^{\frac{p}{p-1},1}}\|\vec{f}\|_{L^{p,\infty}_t\mathcal{Z}_p}\\ &\leq C''A^{-\frac{1}{2}}\|\vec{f}\|_{L^{p,\infty}_t\mathcal{Z}_p}\end{aligned}$$

and

$$\begin{aligned}\|\int_{t-A}^{t} e^{(t-s)\Delta}\vec{f}(s,.)\,ds\|_{\mathcal{Z}_{p,\alpha}} &\leq C\int_{t-A}^{t}\frac{1}{(t-s)^{\alpha/2}}\|f(s,.)\|_{\mathcal{Z}_p}\,ds\\ &\leq C'\|1_{0<t<A}\frac{1}{t^{\alpha/2}}\|_{L^{\frac{p}{p-1},1}}\|\vec{f}\|_{L^{p,\infty}_t\mathcal{Z}_p}\\ &\leq C''A^{\frac{p-1}{p}-\frac{\alpha}{2}}\|\vec{f}\|_{L^{p,\infty}_t\mathcal{Z}_p}\end{aligned}$$

Since

$$(A^{-\frac{1}{2}})^{\frac{p(2-\alpha)-2}{p(3-\alpha)-2}}(A^{\frac{p-1}{p}-\frac{\alpha}{2}})^{\frac{p}{p(3-\alpha)-2}} = 1$$

we find that

$$\vec{v} \in [\mathcal{Z}_{p,\alpha}, L^\infty]_{\frac{p(2-\alpha)-2}{p(3-\alpha)-2},\infty} \subset \mathcal{X}.$$

◇

We may now study the periodic equation and obtain a generalization of Yamazaki's theorem [19] :

Theorem 3
Let $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}_p$ and $\mathcal{Z}_{p,\alpha}$ satisfy assumptions (A1) to (A9). Then there exists a positive constant $\eta_{\mathcal{X},T}$ (which depends on T, $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}_p$ and $\mathcal{Z}_{p,\alpha}$) such that : if $\vec{f}_{\rm per}$ is a time-periodic vector field on $\mathbb{R}\times\mathbb{R}^3$ (with period T) such that

- *the mean value $\vec{f}_0 = \frac{1}{T}\int_0^T \vec{f}_{\rm per}(s,.)\,ds$ belongs to $\dot{B}^{-3}_{\infty,\infty}$ and satisfies $\frac{1}{\Delta}\vec{f}_0 \in \mathcal{X}$ with*
$$\|\frac{1}{\Delta}\mathbb{P}\vec{f}_0\|_{\mathcal{X}} < \eta_{\mathcal{X},T}$$
- *$\vec{f}_{\rm per}$ belongs to $\mathcal{F}^1 = L^1_{\rm per}\mathcal{Z}_1$, or $\mathcal{F}^\infty = L^\infty_t\mathcal{Z}_\infty$ or $\mathcal{F}^p = L^{p,\infty}_{\rm per}\mathcal{Z}_p$ ($1 < p < +\infty$) with*
$$\|\vec{f}\|_{\mathcal{F}^p} < \eta_{\mathcal{X},T}$$

then there exists a time-periodic solution $\vec{u}_{\rm per}$ of the Navier-Stokes problem (5) such that $\vec{u}_{\rm per} \in L^\infty_t \mathcal{X}$.

If moreover $p < 2$ then the solution satisfies $\vec{\nabla} \otimes \vec{u}_{\rm per} \in L^{2,\infty}_{\rm per} \mathcal{X}$. This solution satisfies div $(\vec{u}_{\rm per} \otimes \vec{u}_{\rm per}) = (\vec{u}_{\rm per}.\vec{\nabla})\vec{u}_{\rm per}$.

Proof :
Due to Proposition 6 and to the proof of Theorem 2, we only have to check that $\vec{U}_0 = \int_{-\infty}^t e^{(t-s)\Delta} \mathbb{P} \vec{f}_{\rm per}\, ds$ belongs to $L^\infty_t \mathcal{X}$ (and that $\vec{\nabla} \vec{U}_0$ belongs to $L^{2,\infty}_{\rm per} \mathcal{X}$ if $p < 2$).

We split again $\vec{f}_{\rm per}$ into $\vec{f}_{\rm per} = \vec{f}_0 + \vec{f}_1$ and $\vec{U}_0$ into $\frac{1}{\Delta} \mathbb{P} \vec{f}_0 + \vec{V}_1 + \vec{V}_2$ on $[0, T]$ with $\vec{V}_1 = \int_{-\infty}^t e^{(t-s)\Delta} 1_{[-T,T]}(s) \mathbb{P} \vec{f}_1(s, .)\, ds$. We see easily that, on $[0, T]$, $\frac{1}{\Delta} \mathbb{P} \vec{f}_0 + \vec{V}_2$ and $\partial_j(\frac{1}{\Delta} \mathbb{P} \vec{f}_0 + \vec{V}_2)$ belong to $L^\infty_t \mathcal{X}$. Thus, we just study $\vec{V}_1$. The fact that $\vec{V}_1$ belongs to $L^\infty_t \mathcal{X}$ is given by Proposition 7.

We now study $\partial_j \vec{V}_1$. We have $\mathcal{Z}_{p,\alpha} \subset \dot{B}^{-3+\frac{2}{p}+\alpha}_{\infty,\infty}$, thus we have $\|e^{t\Delta} f\|_{\mathcal{Z}_{p,\alpha}} \leq C \|f\|_{\mathcal{Z}_{p,\alpha}}$ and $\|e^{t\Delta} f\|_\infty \leq C \frac{1}{t^{3/2-\alpha/2-1/p}} \|f\|_{\mathcal{Z}_{p,\alpha}}$. Since $[\mathcal{Z}_{p,\alpha}, L^\infty]_{\frac{p(2-\alpha)-2}{p(3-\alpha)-2},\infty} \subset \mathcal{X}$, we find $\|e^{t\Delta} f\|_{\mathcal{X}} \leq C \frac{1}{t^{1-\alpha/2-1/p}} \|f\|_{\mathcal{Z}_{p,\alpha}}$. Thus, we have

$$\|\partial_j \vec{V}_1(t, x)\|_{\mathcal{X}} \leq C \int_{-\infty}^t \frac{1}{t^{3/2-1/p}} 1_{[-T,T]}(s) \|\vec{f}_1(s, .)\|_{\mathcal{Z}_p}\, ds.$$

If $p < 2$, $3/2 - 1/p < 1$ and $1_{t>0} t^{3/2-1/p} \in L^{\frac{2p}{3p-2},\infty}$. As the convolution maps $L^{\frac{2p}{3p-2},\infty} \times L^{p,\infty}$ to $L^{2,\infty}$, the theorem is proved. ⋄

Examples :
Theorem 3 works in the Lorentz space $L^{3,\infty}$ or more generally on Morrey-Lorentz spaces $\dot{M}^{L^{q,\infty},3}$.

- for $\mathcal{X} = L^{3,\infty}$, take $\mathcal{Y} = L^{3/2,\infty}$ and $\mathcal{Z}_{p,\alpha} = L^{r,\infty}$ with $\frac{1}{r} = 1 - \frac{1}{3}(\alpha + \frac{2}{p})$ (or $\mathcal{Z}_{\infty,0} = L^1$).
- for $\mathcal{X} = \dot{M}^{L^{q,\infty},3}$ with $2 < q < 3$, take $\mathcal{Y} = \dot{M}^{L^{q/2,\infty},3/2}$ and $\mathcal{Z}_{p,\alpha} = \dot{M}^{L^{rq/3,\infty},r}$ with $\frac{1}{r} = 1 - \frac{1}{3}(\alpha + \frac{2}{p})$ and $\alpha + \frac{2}{p} > 3 - q$.

5 Sobolev estimates and the Navier–Stokes problem.

In this section, we shall study solutions which belongs to $L^\infty_t \dot{H}^{1/2} \cap L^2_{\rm per} \dot{H}^{3/2}$. Following the formalism of Kyed [11], we get easily a theorem in Sobolev norms :

Theorem 4
There exists a positive constant η *such that : if* $\vec{f}_{\rm per}$ *is a time-periodic vector field on* $\mathbb{R}\times\mathbb{R}^3$ *(with period* T*) such that*

- *the mean value* $\vec{f}_0 = \frac{1}{T}\int_0^T \vec{f}_{\rm per}(s,.)\,ds$ *belongs to* $\dot H^{-3/2}$ *and satisfies*
$$\|\mathbb{P}\vec{f}_0\|_{\dot H^{-3/2}} < \eta$$
- $\vec{f}_{\rm per}$ *belongs to* $L^2_{\rm per}\dot H^{-1/2}$ *with*
$$\|\vec{f}_{\rm per}\|_{L^2_{\rm per}\dot H^{-1/2}} < \eta$$

then there exists a time-periodic solution $\vec{u}_{\rm per}$ *of the Navier-Stokes problem (5) such that* $\vec{u}_{\rm per} \in L^\infty_t \dot H^{1/2} \cap L^2_{\rm per}\dot H^{3/2}$.

Proof :
We first study $\vec{U}_0 = \int_{-\infty}^t e^{(t-s)\Delta}\mathbb{P}\vec{f}\,ds$. We expand $\mathbb{P}\vec{f}$ as a time-Fourier series
$$\mathbb{P}\vec{f} = \sum_{k\in\mathbb{Z}} \vec{g}_k(x) e^{\frac{2\pi}{T}ikt}.$$
We have
$$\int_0^T \|\mathbb{P}\vec{f}\|^2_{\dot H^{-1/2}}\,dt = T\sum_{k\in\mathbb{Z}} \|\vec{g}_k\|^2_{\dot H^{-1/2}}.$$
The Fourier expansion of $\vec{U}_0$ is
$$\vec{U}_0 = \sum_{k\in\mathbb{Z}} \vec{W}_k(x) e^{\frac{2\pi}{T}ikt}.$$
with
$$\vec{W}_k = \frac{1}{ik\frac{2\pi}{T} - \Delta}\mathbb{P}\vec{g}_k.$$
We have $\|\vec{W}_k\|_{\dot H^{3/2}} \le \|\vec{g}_k\|_{\dot H^{1/2}}$, and thus $\vec{U}_0 \in L^2_{\rm per}\dot H^{3/2}$. Moreover, $\vec{g}_0 = \mathbb{P}\vec{f}_0 \in \dot H^{-3/2}$ so that $\vec{W}_0 \in \dot H^{1/2}$. Let $\vec{\Omega}_k$ be the Fourier transform of $\vec{g}_k$. We have :
$$\begin{aligned}(2\pi)^3\|\vec{U}_0(t,.) - \vec{W}_0\|^2_{\dot H^{1/2}} &= \int |\xi| \left|\sum_{k\neq 0} \frac{1}{ik\frac{2\pi}{T} + |\xi|^2}\vec{\Omega}_k(\xi) e^{\frac{2\pi}{T}ikt}\right|^2 d\xi \\ &\le \int |\xi|\,(\sum_{k\neq 0}\frac{1}{k^2\frac{4\pi^2}{T^2} + |\xi|^4})(\sum_{k\neq 0}|\vec{\Omega}_k(\xi)|^2)\,d\xi\end{aligned}$$

If $T|\xi|^2 \leq 1$, we write

$$\sum_{k\neq 0} \frac{1}{k^2 \frac{4\pi^2}{T^2} + |\xi|^4} \leq \frac{T^2}{4\pi^2} \sum_{k\neq 0} \frac{1}{k^2} = \frac{T^2}{12} \leq \frac{T}{12|\xi|^2}.$$

If $T|\xi|^2 > 1$, we write

$$\sum_{k\neq 0} \frac{1}{k^2 \frac{4\pi^2}{T^2} + |\xi|^4} \leq 2(\sum_{1\leq k \leq 2T|\xi|^2} \frac{1}{|\xi|^4} + \sum_{k > 2T|\xi|^2} \frac{1}{k^2 \frac{4\pi^2}{T^2}}) \leq (4 + \frac{1}{2\pi^2}) \frac{T}{|\xi|^2}$$

Thus, we find that $\vec{U}_0 \in L^\infty_t \dot{H}^{1/2}$.

It is now easy to check that the bilinear operator B is bounded on $E = L^\infty_t \dot{H}^{1/2} \cap L^2_{\rm per} \dot{H}^{3/2}$. Indeed, we have, for $\vec{U}$ and $\vec{V}$ in E,

$$\|\vec{U}(t,.)\otimes\vec{V}(t,.)\|_{\dot{H}^{1/2}} \leq C(\|\vec{U}(t,.)\|_{\dot{H}^{1/2}}\|\vec{V}(t,.)\|_{\dot{H}^{3/2}} + \|\vec{V}(t,.)\|_{\dot{H}^{1/2}}\|\vec{U}(t,.)\|_{\dot{H}^{3/2}})$$

and

$$\|\vec{U}(t,.)\otimes\vec{V}(t,.)\|_{\dot{H}^{-1/2}} \leq C\|\vec{U}(t,.)\|_{\dot{H}^{1/2}}\|\vec{V}(t,.)\|_{\dot{H}^{1/2}}$$

so that $\vec{F} = \text{div } (\vec{U}\otimes\vec{V})$ satisfies $\vec{F} \in L^2_{per}\dot{H}^{-1/2}$ and $\int_0^T \vec{F}(s,.)\, ds \in \dot{H}^{-3/2}$. The proof we gave on $\vec{U}_0$ gives us as well that $B(\vec{U},\vec{V}) \in E$. $\diamond$

Sometimes, it is convenient to have solutions in the smaller space $L^\infty_t \dot{H}^{1/2} \cap L^2_{\rm per} \dot{H}^{3/2} \cap L^2_{\rm per} L^\infty$ [5]. Let $\tilde{L}^2_{\rm per} \dot{B}^{-1/2}_{2,1}$ be the space of periodic distributions $f \in L^2_{\rm per} \dot{H}^{-1/2}$ such that their Littlewood–Paley decompositions $\vec{f} = \sum_{j\in\mathbb{Z}} \Delta_j f$ satisfy

$$\sum_{j\in\mathbb{Z}} 2^{-j/2} \sqrt{\int_0^T \|\Delta_j f\|_2^2} < +\infty.$$

Such a space was first introduced by Chemin and Lerner [3].

Theorem 5
There exists a positive constant η such that : if $\vec{f}_{\rm per}$ is a time-periodic vector field on $\mathbb{R}\times\mathbb{R}^3$ (with period T) such that

- *the mean value $\vec{f}_0 = \frac{1}{T}\int_0^T \vec{f}_{\rm per}(s,.)\, ds$ belongs to $\dot{H}^{-3/2}$ and satisfies*

$$\|\mathbb{P}\vec{f}_0\|_{\dot{H}^{-3/2}} < \eta$$

- *$\vec{f}_{\rm per} = \vec{g}_1 + \vec{g}_2$ with $\vec{g}_1 \in L^1_{\rm per}\dot{H}^{1/2}$, $\vec{g}_2 \in \tilde{L}^2_{\rm per}\dot{B}^{-1/2}_{2,1}$ and*

$$\|\vec{g}_1\|_{L^1_{\rm per}\dot{H}^{1/2}} + \|\vec{g}_2\|_{\tilde{L}^2_{\rm per}\dot{B}^{-1/2}_{2,1}} < \eta$$

then there exists a time-periodic solution $\vec{u}_{\rm per}$ of the Navier-Stokes problem (5) such that $\vec{u}_{\rm per} \in L^\infty_t \dot H^{1/2} \cap L^2_{\rm per}\dot H^{3/2} \cap L^2_{\rm per} L^\infty$.

Proof :
We first check that $\vec{U}_0$ belongs to $L^\infty_t \dot H^{1/2} \cap L^2_{\rm per}\dot H^{3/2} \cap L^2_{\rm per} L^\infty$. We call $\vec{h}_1$ the fluctuation of $\vec{g}_1$, $\vec{h}_2$ the fluctuation of $\vec{g}_2$, so that $\vec{U}_0 = \frac{1}{\Delta}\vec{f}_0 + \vec{W}_1 + \vec{W}_2$ where $\vec{W}_i = \int_{-\infty}^t e^{(t-s)\Delta}\mathbb{P}\vec{h}_i\, ds$. Since $\vec{f}_0 \in \dot H^{-3/2} \cap \dot B^{-1/2}_{2,1}$, we have $\frac{1}{\Delta}\vec{f}_0 \in \dot H^{1/2} \cap \dot B^{3/2}_{2,1} \subset \dot H^{1/2} \cap \dot H^{3/2} \cap L^\infty$. The case of $\vec{W}_2$ is easy as well : from the proof of Theorem 4, we know that

$$\| \int_{-\infty}^t e^{(t-s)\Delta}\mathbb{P}\Delta_j \vec{h}_2\, ds\|_{L^\infty_t \dot H^{1/2}\cap L^2_{\rm per}\dot H^{3/2}} \leq C\|\Delta_j \vec{h}_2\|_{L^2_{\rm per}\dot H^{-1/2}} \leq C' 2^{-j/2}\|\Delta_j\vec{h}_2\|_{L^2_{\rm per}L^2}$$

As

$$2^{3j/2}\|\Delta_j \vec{W}_2\ \|_{L^2_{\rm per}L^2} \leq C\| \int_{-\infty}^t e^{(t-s)\Delta}\mathbb{P}\Delta_j\vec{h}_2\, ds\|_{L^\infty_t\dot H^{1/2}\cap L^2_{\rm per}\dot H^{3/2}},$$

we get that $\vec{W}_2 \in L^\infty_t \dot H^{1/2} \cap \tilde L^2_{\rm per}\dot B^{3/2}_{2,1} \subset L^\infty_t\dot H^{1/2}\cap L^2_{\rm per}\dot H^{3/2}\cap L^2_{\rm per}L^\infty$.

Finally, introducing the periodic distributions $\vec{F} = \int_0^t \vec{h}_1\, ds$, $\vec{F}_0 = \frac{1}{T}\int_0^T \vec{F}\, ds$ and $\vec{G} = \int_0^t (\vec{F}(s,.) - \vec{F}_0)\, ds$, we write, on $[0,T]$, $\vec{W}_1$ as $\vec{W}_1 = \vec{Z}_1 + \vec{Z}_2$ with

$$\vec{Z}_1 = \int_{-\infty}^t e^{(t-s)\Delta}\mathbb{P}(1_{[-T,T]}(s)\vec{h}_1(s,.))\, ds$$

and

$$\vec{Z}_2 = -\Delta e^{(t+T)\Delta}\vec{F}_0 + \int_{-\infty}^{-T} \Delta^2 e^{(t-s)\Delta}\, ds$$

It is easy to check that $\vec{Z}_2$ belongs to $L^\infty([0,T], \dot H^{1/2}\cap\dot H^{3/2}\cap L^\infty)$. Moreover we have for $\vec\gamma$ defined on $[0,T]\times\mathbb{R}^3$

$$\int_0^T \int \vec\gamma(t,x).\vec{Z}_1(t,x)\, dt\, dx = \int_{-T}^T \left(\int_{\max(0,-s)}^{T-s} \int_{\mathbb{R}^3} \vec\gamma(\theta+s).e^{t\Delta}\mathbb{P}\vec{h}_1(s,.)\, dx\, d\theta \right)\, ds$$

When $u_0 \in \dot H^{1/2} \subset \dot B^{-1}_{\infty,2}$, we have $1_{\theta>0}e^{\theta\Delta}\vec{u}_0 \in L^\infty_t\dot H^{1/2}\cap L^2_t\dot H^{3/2}\cap L^2_t L^\infty$ so that, for

$$I(s) = \int_{\max(0,-s)}^{T-s}\int_{\mathbb{R}^3} \vec\gamma(\theta+s).e^{t\Delta}\mathbb{P}\vec{h}_1(s,.)\, dx\, d\theta,$$

we have

$$|I(s)| \leq C\|\vec{h}_1(s,.)\|_{\dot H^{1/2}} \min(\|\vec\gamma\|_{L^1_t\dot H^{-1/2}}, \|\vec\gamma\|_{L^2_t\dot H^{-3/2}}, \|\vec\gamma\|_{L^2_t L^\infty})$$

and we find by duality that $\vec{Z}_1$ belongs to $L^\infty([0,T],\dot H^{1/2})\cap L^2([0,T],\dot H^{3/2}\cap L^\infty)$. Thus, we have $\vec{U}_0 \in L^\infty_t \dot H^{1/2}\cap L^2_{\rm per}\dot H^{3/2}\cap L^2_{\rm per}L^\infty$.

Again, it is then easy to conclude by checking that the bilinear operator B is bounded on $E = L^\infty_t \dot H^{1/2}\cap L^2_{\rm per}\dot H^{3/2}\cap L^2_{\rm per}L^\infty$. Indeed, we have, for $\vec{U}$ and $\vec{V}$ in E,

$$\|\vec{U}(t,.)\otimes\vec{V}(t,.)\|_{\dot H^{3/2}} \le C(\|\vec{U}(t,.)\|_\infty\|\vec{V}(t,.)\|_{\dot H^{3/2}} + \|\vec{V}(t,.)\|_\infty\|\vec{U}(t,.)\|_{\dot H^{3/2}})$$

and

$$\|\vec{U}(t,.)\otimes\vec{V}(t,.)\|_{\dot H^{-1/2}} \le C\|\vec{U}(t,.)\|_{\dot H^{1/2}}\|\vec{V}(t,.)\|_{\dot H^{1/2}}$$

so that $\vec{F} = \text{ div }(\vec{U}\otimes\vec{V})$ satisfies $\vec{F}\in L^1_{per}\dot H^{1/2}$ and $\int_0^T \vec{F}(s,.)\,ds \in \dot H^{-3/2}$. The proof we gave on $\vec{U}_0$ gives us as well that $B(\vec{U},\vec{V})\in E$. $\diamond$

6 Asymptotic stability.

We shall now discuss the Cauchy initial value problem

More precisely, we consider the equations on $(0,+\infty)\times\mathbb{R}^3$

$$\begin{cases} \partial_t\vec{u} + \text{ div }(\vec{u}\otimes\vec{u}) = \Delta\vec{u} + \vec{f}_{\rm per} - \vec{\nabla}p \\ \text{div }\vec{u} = 0 \\ \vec{u}(0,.) = \vec{u}_0 \end{cases} \tag{8}$$

and prove a result on the asymptotic stability of the periodic solutions (when the force is small enough and the initial value small enough).

The proof will be based on ideas of Yamazaki [19] in the case of $\vec{u}_0 \in BMO^{-1}$ (where the space BMO^{-1} is the space introduced by Koch and Tataru [9] as the largest critical space where to search for initial values for getting mild solutions) and on a recent preprint of Karch, Pilarczyk and Schonbek [6] in the case $\vec{u}_0\in L^2$.

We shall use the following spaces of functions on $(0,+\infty)$ for estimating the stability of the time-periodic solutions :

- the Koch–Tataru space E_{KT} : $f\in E_{KT}$ if f satisfies

$$\sup_{t>0}\sqrt{t}\|f(t,.)\|_\infty < +\infty \text{ and } \sup_{t>0,x_0\in\mathbb{R}^3}\frac{1}{t^{3/2}}\|1_{[0,t]}(s)1_{B(x_0,\sqrt{t})}(y)f(s,y)\|_{L^2_{t,x}} < +\infty.$$

- the Morrey–Lorentz space E_{ML} : $f\in E_{ML}$ if f satisfies

$$\sup_{t>0} t^{1/4}\|f(t,.)\|_{\dot M^{L^{4,\infty},6}} < +\infty$$

- the Leray space E_{JL} : $f \in E_{JL}$ if f satisfies

$$f \in L^\infty_t L^2 \cap L^2_t \dot H^1 \text{ and } \lim_{t\to+\infty} \|f(t,.)\|_2 = 0.$$

We have the following obvious property : if f belongs to $E_{KT} + E_{ML} + E_{JL}$ and if Ω is a bounded domain of $\mathbb{R}^3$, then $\lim_{t\to+\infty} \int_\Omega |f(t,x)|^2\,dx = 0$.

Theorem 6
Let $\vec f_{\rm per}$ be a time-periodic force with mean value $\vec f_0$ and fluctuation $\vec f_1$. small enough in the space $\mathbb{X}$, where $\mathbb{X}$ is one of the following spaces :

- *for some $\mathcal{X}$ and $\mathcal{Y}$ satisfying (A1) to (A4), $\vec f_{\rm per} \in \mathbb{X}$ if and only if $\vec f_{\rm per} \in L^1_{\rm per}\mathcal{X}$ and $\frac{1}{\Delta}\vec f_0 \in \mathcal{X}$*
- *for some $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}_p$ and $\mathcal{Z}_{p,\alpha}$, satisfying (A1) to (A9), $\vec f_{\rm per} \in \mathbb{X}$ if and only if $\vec f_{\rm per} \in \mathcal{F}_p$ and $\frac{1}{\Delta}\vec f_0 \in \mathcal{X}$, where $\mathcal{F}_1 = L^1_{\rm per}\mathcal{Z}_1$, $\mathcal{F}_\infty = L^\infty_t \mathcal{Z}_\infty$, $\mathcal{F}_p = L^{p,\infty}_{\rm per}\mathcal{Z}_p$ for $1 < p < +\infty$*
- *for $\mathcal{X} = \dot H^{1/2}$, $\vec f_{\rm per} \in \mathbb{X}$ if and only if $\vec f_{\rm per} \in L^1_{\rm per}\dot H^{1/2} + L^2_{\rm per}\dot H^{-1/2}$ and $\frac{1}{\Delta}\vec f_0 \in \dot H^{1/2}$.*

Then

- *(A) There exists a time-periodic solution $\vec u_{\rm per} \in \mathbb{Y} = L^\infty_t\mathcal{X}$ of the Navier-Stokes problem (5).*
- *(B) If $\vec u_0 \in \mathcal{X}$ is small enough, then there exists a global solution $\vec u \in L^\infty((0,+\infty),\mathcal{X})$ of the initial value problem (8). Moreover, we have $\vec u - \vec u_{\rm per} \in E_{ML}$, so that $\vec u$ converges asymptotically to $\vec u_{\rm per}$ when t goes to $+\infty$.*
- *(C) If $\vec u_0 \in BMO^{-1}$ is small enough, then there exists a global solution $\vec u$ of the initial value problem (8), such that we have $\vec u - \vec u_{\rm per} \in E_{ML} + E_{KT}$. Thus, $\vec u$ converges asymptotically to $\vec u_{\rm per}$ when t goes to $+\infty$.*
- *(D) If $\vec u_0 \in L^2$, then there exists a global solution $\vec u$ of the initial value problem (8), such that we have $\vec u - \vec u_{\rm per} \in E_{ML} + E_{JL}$. Thus, $\vec u$ converges asymptotically to $\vec u_{\rm per}$ when t goes to $+\infty$.*

Proof :
Point (A) has been proved in the previous sections.

In order to prove point (B), we construct the solution $\vec{u}$ as the limit of an iterative process $\vec{u} = \lim_{n\to 0} \vec{V}_n$ with $\vec{V}_0 = e^{t\Delta}\vec{u}_0 + \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{f}_{\text{per}}(s,.)\,ds$ and $\vec{V}_{n+1} = \vec{V}_0 - B_0(\vec{V}_n, \vec{V}_n)$, where

$$B_0(\vec{U}, \vec{W}) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\text{div }(\vec{U}\otimes\vec{V})\,ds.$$

From Propositions 4 and 6 and from Theorem 4, we see that B_0 is bounded on $\mathcal{E}$, $L_t^\infty\mathcal{X}$ or $L_t^\infty\dot{H}^{1/2}\cap L_t^2\dot{H}^{3/2}$ and that $\vec{V}_0$ belongs to $\mathcal{E}$, $L_t^\infty\mathcal{X}$ or $L_t^\infty\dot{H}^{1/2}\cap L_t^2\dot{H}^{3/2}$. This proves the existence of a solution $\vec{u}$ in $L_t^\infty\mathcal{X}$ when the force and the initial value are small enough.

To check the asymptotic stability, we write $\vec{u} = \vec{u}_{\text{per}} + \vec{w}$. $\vec{w}$ is the solution of the initial value problem

$$\begin{cases} \partial_t\vec{w} + \text{ div }(\vec{w}\otimes\vec{w}) = \Delta\vec{w} - \text{ div }(\vec{w}\otimes\vec{u}_{\text{per}}) - \text{ div }(\vec{u}_{\text{per}}\otimes\vec{w}) - \vec{\nabla}q \\ \hfill \text{div }\vec{w} = 0 \\ \hfill \vec{w}(0,.) = \vec{u}_0 - \vec{u}_{\text{per}}(0,.) \end{cases} \quad (9)$$

thus we have $\vec{w} = \lim_{n\to+\infty}\vec{W}_n$ with $\vec{W}_0 = e^{t\Delta}(\vec{u}_0 - \vec{u}_{\text{per}}(0,.))$ and $\vec{W}_{n+1} = \vec{W}_0 - B_0(\vec{W}_n, \vec{W}_n) - B(\vec{u}_{\text{per}}, \vec{W}_n) - B_0(\vec{W}_n, \vec{u}_{\text{per}})$. As $\mathcal{X}\subset\mathcal{V}^{2,1}\subset\dot{M}^{2,3}\subset\dot{B}^{-1}_{\infty,\infty}$, we have that $e^{t\Delta}$ maps $\mathcal{X}$ to $[\dot{M}^{2,3}, L^\infty]_{1/2,\infty}$ with an operator norm which is $O(t^{-1/4})$; since $[\dot{M}^{2,3}, L^\infty]_{1/2,\infty}\subset\dot{M}^{L^{4,\infty},6}$, we find that $\vec{W}_0\in E_{ML}$. Moreover, B_0 maps $E_{ML}\times E_{ML}$, $E_{ML}\times L_t^\infty\dot{M}^{2,3}$ and $L_t^\infty\dot{M}^{2,3}\times E_{ML}$ to E_{ML}. This gives that (when the data are small) the iterates $\vec{W}_n$ converge in the norm of E_{ML}, so that $\vec{u} - \vec{u}_{\text{per}}\in E_{ML}$.

In particular, we may consider $\vec{u}_0 = 0$, and get a solution $\vec{u}^{(0)}$ of the Cauchy problem. Thus, when we consider a Cauchy initial value problem for an initial data $\vec{u}_0$, we may write the solution $\vec{u}$ as a pertirbation of $\vec{u}^{(0)}$. Writing $\vec{u} = \vec{u}^{(0)} + \vec{w}$, we find that $\vec{w}$ is a solution of the initial value problem

$$\begin{cases} \partial_t\vec{w} + \text{ div }(\vec{w}\otimes\vec{w}) = \Delta\vec{w} - \text{ div }(\vec{w}\otimes\vec{u}^{(0)}) - \text{ div }(\vec{u}^{(0)}\otimes\vec{w}) - \vec{\nabla}q \\ \hfill \text{div }\vec{w} = 0 \\ \hfill \vec{w}(0,.) = \vec{u}_0 \end{cases} \quad (10)$$

or equivalently of

$$\vec{w} = e^{t\Delta}\vec{u}_0 - B_0(\vec{w}, \vec{w}) - B_0(\vec{u}^{(0)}, \vec{w}) - B_0(\vec{w}, \vec{u}^{(0)}).$$

If $\vec{u}_0$ belongs to BMO^{-1}, $e^{t\Delta}\vec{u}_0$ belongs to E_{KT}. Koch and Tataru [9] proved that B_0 maps $E_{KT}\times E_{KT}$ to E_{KT}. On the other hand, B_0 maps $E_{KT}\times L_t^\infty\dot{M}^{2,3}$, $L_t^\infty\times\dot{M}^{2,3}\times E_{KT}$, $E_{ML}\times E_{ML}$, $E_{ML}\times L_t^\infty\dot{M}^{2,3}$ and $L_t^\infty\dot{M}^{2,3}\times E_{ML}$

to E_{ML}. This gives that (when the data are small) we find a solution $\vec{w}$ of equations (10) with $\vec{w} \in E_{KT} + E_{ML}$.

Finally, the system (10) with $\vec{u}^{(0)}$ small in $L^\infty_t \mathcal{V}^{1,2}$ and with initial value $\vec{u}_0 \in L^2$ has been studied by Karch, Pilarczyk and Schonbek [6]. They proved the existence of weak solutions $\vec{w} \in E_{JL}$. $\diamond$

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