Parabolic Morrey spaces and mild solutions to Navier–Stokes equations.

An interesting answer through a silly method to a stupid question.

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Abstract

We present a theory of mild solutions for the Navier–Stokes equations in a (maximal) lattice Banach space.

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1 The stupid question.

Our question concerns the search of mild solutions for the Navier–Stokes problem. More precisely, let us consider the following Cauchy initial value problem for the Navier–Stokes equations on the whole space and with no external forces (and with viscosity taken equal to 1) :

$$\begin{cases} \partial_t \vec{u} = \Delta \vec{u} - (\vec{u}.\vec{\nabla})\vec{u} - \vec{\nabla} p \\ \vec{u}(0,x) = \vec{u}_0(x) \\ \text{div}\vec{u} = 0 \end{cases} \tag{1}$$

When looking for a mild solution, one rewrites the problem as a fixed point problem of an integro–differential transform

$$\vec{u} = e^{t\Delta}\vec{u}_0 - B(\vec{u},\vec{u}) \tag{2}$$

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where the bilinear transform B is defined as

$$B(\vec{u},\vec{v}) = \int_0^t e^{(t-s)\Delta}\mathbb{O}(\vec{u}(s,.)\otimes\vec{v}(s,.))\,ds. \tag{3}$$

$\mathbb{O}$ is the Oseen operator mapping matrix functions $F = (F_{ij})$ to vector functions $\vec{H} = (H_k)$ through the formula

$$H_k = \sum_{i,j}\mathbb{O}_{ijk}F_{ij} = \sum_{i,j}(\delta_{j,k}\partial_i - \frac{1}{\Delta}\partial_i\partial_j\partial_k)F_{i,j} \tag{4}$$

Mild solutions are then searched through Picard's iterative scheme : starting from $\vec{U}_0(t,x) = e^{t\Delta}\vec{u}_0$ and defining $\vec{U}_{n+1} = \vec{U}_0 - B(\vec{U}_n,\vec{U}_n)$, check whether the sequence $\vec{U}_n$ converges to a limit $\vec{u}$.

Our (stupid) question is then the following one :

Question 1
Which is the largest space X such that $\|\vec{u}_0\|_X$ small enough implies that $\vec{U}_n$ converge to a global mild solution?

To the unaware reader, the question might appear as sensible. However, it is a stupid question nowadays, since the answers has been known for fifteen years (Koch–Tataru theorem (2001) [6]) :

$$X = \mathrm{BMO}^{-1}.$$

If we would like to alleviate the suspicion that we are dealing with some uninteresting problem, one may consider the same problem for the generalized Navier–Stokes problem where we replace the Laplacian operator by a fractional Laplacian operator :

$$\begin{cases} \partial_t\vec{u} = -(-\Delta)^{\alpha/2}\vec{u} - (\vec{u}.\vec{\nabla})\vec{u} - \vec{\nabla}p \\ \vec{u}(0,x) = \vec{u}_0(x) \\ \mathrm{div}\vec{u} = 0 \end{cases} \tag{5}$$

where $1 < \alpha$

The answer to the question is then :

- $\alpha = 2$: $X = \mathrm{BMO}^{-1}$ [based on integration by parts]
- $1 < \alpha < 2$: $X = \dot{B}^{1-\alpha}_{\infty,\infty}$ [no need to integrate by parts]
- $\alpha > 2$: unknown (at least, to me) [integration by parts does not work]

2 The silly method.

Now, in order to try and provide an answer to Question 1, we are going to introduce a method that clearly cannot provide optimal answers (this is why I shall call it a silly method).

Let us recall that we have transformed the differential equation (1) into an (integro-)differential equation (2). Let $\mathbb{K}(t,x)$ be the integral kernel of the operator matrix $e^{t\Delta}\mathbb{O}$, so that the equation to be solved reads as

$$\vec{u} = e^{t\Delta}\vec{u}_0 + \int_0^t \int \mathbb{K}(t-s, x-y)(\vec{u}(s,y)\otimes\vec{u}(s,y))\, dy\, ds. \tag{6}$$

As it is an integral equation, we want to use basic tools of integration such as Fatou's lemma, monotone convergence or dominated convergence. This is much easier when the integrand is nonnegative. Thus, we shall replace the equation (6) by a superequation :

$$U(t,x) = |e^{t\Delta}\vec{u}_0| + \int_0^t \int |\mathbb{K}(t-s, x-y)|\, U^2(s,y))\, dy\, ds. \tag{7}$$

While we gain on simplicity for the integral term to be dealt with, we definitely loose the main tool we have to control the solutions of the Navier–Stokes equations : we destroy any hope to use the dissipation expressed by the Leray energy inequality.

However, nonnegativity of the kernel is good, but we could have better : symmetry. Thus, we shall use a further generalization of the equation, and consider the equation :

$$U(t,x) = 1_{t>0}|e^{t\Delta}\vec{u}_0| + \int_{s=-\infty}^{s=+\infty} \int |\mathbb{K}(|t-s|, x-y)|\, U^2(s,y))\, dy\, ds. \tag{8}$$

The last bold step toward simplification will be to replace the kernel $\mathbb{K}$ by a simpler kernel. A well-known estimate states that we have

$$|\mathbb{K}(t,x)| \le C_0 \frac{1}{t^2+|x|^4} \tag{9}$$

for some positive constant C_0. The equation we shall consider is then

$$U(t,x) = 1_{t>0}|e^{t\Delta}\vec{u}_0| + C_0 \int_{\mathbb{R}} \int \frac{1}{(t-s)^2+|x-y|^4}\, U^2(s,y))\, dy\, ds. \tag{10}$$

More precisely, if $W_0(t,x)$ is defined on $\mathbb{R}\times\mathbb{R}^3$ is such that the iterative sequence defined by induction from W_0 through

$$W_{n+1}(t,x) = W_0(t,x) + C_0 \int_{\mathbb{R}} \int \frac{1}{(t-s)^2+|x-y|^4}\, W_n^2(s,y))\, dy\, ds \tag{11}$$

satisfies

$$\sup_{n\in\mathbb{N}} W_n(t,x) < \infty \quad \text{a.e.} \tag{12}$$

on $\mathbb{R}\times\mathbb{R}^3$ then we have the following consequences :

- $W(t,x) = \sup_{n\in\mathbb{N}} W_n(t,x)$ is a locally integrable function which satisfies

$$W(t,x) = W_0(t,x) + C_0 \int_{\mathbb{R}} \int \frac{1}{(t-s)^2 + |x-y|^4}\, W^2(s,y)\, dy\, ds. \tag{13}$$

- if $1_{t>0}|e^{t\Delta}\vec{u}_0| \leq W_0(t,x)$, $\vec{U}_0 = 1_{t>0}e^{t\Delta}\vec{u}_0$ and $\vec{U}_{n+1} = \vec{U}_0 - 1_{t>0}B(\vec{U}_n,\vec{U}_n)$ then we have

$$|\vec{t}\vec{U}_{n+1}(t,x) - \vec{U}_n(t,x)| \leq W_{n+1}(t,x) - W_n(t,x) \tag{14}$$

so that

$$|\vec{U}_0(t,x)| + \sum_{n\in\mathbb{N}} |\vec{U}_{n+1}(t,x) - \vec{U}_n(t,x)| \leq W(t,x) \tag{15}$$

so that we find a mild solution $\vec{u}$ of equation (2).

Thus, we are lead to study the following questions :

Question 2
For which functions $W_0 \geq 0$ can we say that, for $\epsilon > 0$ small enough, we have a solution to the integral equation (13) for ϵW_0

$$W_\epsilon = \epsilon W_0 + \int_{\mathbb{R}} \int_{\mathbb{R}^3} \frac{C_0}{(t-s)^2 + |x-y|^4} W_\epsilon^2(s,y)\, ds\, dy?$$

Question 3
For which spaces of good initial values for the Navier–Stokes equations can we say that $W_0(t,x) = 1_{t>0}|e^{t,\Delta}\vec{u}_0|$ will satisfy Question 2?

More precisely, how much did we loose by changing Question 1 into Question 2?

3 Elliptic intermezzo.

Before considering Question 2, we recall some basic facts involving integral equations with symmetric non-negative kernels [7].

We thus look at the general integral equation

$$f(x) = f_0(x) + \int_X K(x,y) f^2(y)\ d\mu(y) \tag{16}$$

where μ is a non-negative σ-finite measure on a space X ($X = \cup_{n\mathbb{N}} Y_n$ with $\mu(Y_n) < +\infty$), and K is a positive measurable function on $X \times X$: $K(x,y) > 0$ almost everywhere. We shall make a stronger assumption on K : there exists a sequence X_n of measurable subsets of X such that $X = \cup_{n\in\mathbb{N}} X_n$ and

$$\int_{X_n}\int_{X_n} \frac{d\mu(x)\ d\mu(y)}{K(x,y)} < +\infty. \tag{17}$$

Obviously, if f_0 is non-negative and f is an (almost everywhere finite) non-negative measurable solution of equation (16), then we have $0 \le f_0 \le f$ and

$$\int_X K(x,y) f^2(y)\ d\mu(y) \le f(x) \quad \text{a.e..}$$

Conversely, if $0 \le f_0 < \frac{1}{4}\Omega$, with $\int_X K(x,y)\Omega^2(y)\ d\mu(y) \le \Omega(x)$ a.e., then there exists an (almost everywhere finite) non-negative measurable solution of equation (16).

This gives the space where to search for solutions of equation (16) :

Proposition 1
Let $\mathcal{E}_K$ be the space of measurable functions f on X such that there exists $\lambda \ge 0$ and a measurable non-negative function Ω such that $|f(x)| \le \lambda\Omega$ almost everywhere and $\int_X K(x,y)\Omega^2(y)\ d\mu(y) \le \Omega(x)$ a.e.. Then :

- *$\mathcal{E}_K$ is a linear space*
- *$\|f\|_K = \inf\{\lambda\ /\ \exists\Omega \ge 0\quad |f| \le \lambda\Omega$ and $\int_X K(x,y)\Omega^2(y)\ d\mu(y) \le \Omega(x)$ a.e.$\}$ is a semi-norm on $\mathcal{E}_K$*
- *$\|f\|_K = 0 \Leftrightarrow f = 0$ almost everywhere*
- *The normed linear space E_K (obtained from $\mathcal{E}_K$ by quotienting with the relationship $f \sim g \Leftrightarrow f = g$ a.e.) is a Banach space.*
- *If $f_0 \in \mathcal{E}_K$ is non-negative and satisfies $\|f_0\|_K < \frac{1}{4}$, then equation (16) has a non-negative solution $f \in \mathcal{E}_K$.*

Our first example will be the elliptic non-linear equation on $\mathbb{R}^d$ ($d \ge 3$)

$$-\Delta u = (-\Delta)^{1/2} u^2 - \Delta V$$

This can be rewritten as

$$u = V + \mathcal{I}_1(u^2) \tag{18}$$

where the Riesz potential $\mathcal{I}_1$ is given by

$$\mathcal{I}_1 f(x) = \frac{1}{(-\Delta)^{1/2}} f(x) = \int_{\mathbb{R}^d} \frac{C_1}{|x-y|^{d-1}} f(y)\ dy.$$

The answer to Question 2 for equation (18) is well known [9] :

Theorem 1 (Maz'ya and Verbitsky 1995)
Let $V \geq 0$. Then the following assertions are equivalent :

1. *for $\epsilon > 0$ small enough, we have a solution to the equation $u_\epsilon = \epsilon V + \mathcal{I}_1(u_\epsilon^2)$*

2. *V satisfies the inequality :*

$$\exists C \geq 0\ \forall f \in L^2\ \int_{\mathbb{R}^d} V^2(x)(\mathcal{I}_1 f(x))^2\ dx \leq C \int_{\mathbb{R}^d} f^2(x)\ dx$$

3. *V is a multiplier from the homogeneous Sobolev space $\dot{H}^1$ to L^2 :*

$$\exists C \geq 0\ \forall f \in \dot{H}^1\ \int_{\mathbb{R}^d} V^2(x) f^2(x)\ dx \leq C \int_{\mathbb{R}^d} |\vec{\nabla} f(x)|^2\ dx$$

Thus, we can see that the answer to Question 2 is far from being obvious. The maximal functional space where to look for solutions is no classical space, it is the space of singular multipliers $\mathcal{V} = \mathcal{M}(\dot{H}^1 \mapsto L^2)$ from $\dot{H}^1$ to L^2.

If we want to deal with some more amenable spaces, one can use the Fefferman–Phong inequality [3] that relates the multiplier space to Morrey spaces. For $1 < p \leq q < +\infty$, let us define the (homogeneous) Morrey space $\dot{M}^{p,q}$ in the following way : $f \in \dot{M}^{p,q}$ $(1 < p \leq q < +\infty)$ if $\sup_{R>0, x\in\mathbb{R}^d} R^{d(\frac{p}{q}-1)} \int_{|x-y|<R} |f(y)|^p\ dy < +\infty$. Then we have :

Theorem 2 (Fefferman–Phong 1983)
For $2 < p \leq d$, we have

$$\dot{M}^{p,d} \subset \mathcal{V} \subset \dot{M}^{2,d}$$

Maz'ya and Verbitsky(s theorem has been generalized to spaces of homogeneous type by Kalton and Verbitsky [4] :

Theorem 3 (Kalton and Verbitsky 1999)
Let (X, δ, μ) be a space of homogeneous type :

- *for all $x, y \in X$, $\delta(x,y) \geq 0$*
- $\delta(x,y) = \delta(y,x)$
- $\delta(x,y) = 0 \Leftrightarrow x = y$
- *there is a positive constant κ such that :*

$$\text{for all } x, y, z \in X, \delta(x,y) \leq \kappa(\delta(x,z) + \delta(z,y)) \tag{19}$$

- *there exists postive A, B and Q which satisfy :*

$$\text{for all } x \in X, \text{ for all } r > 0, Ar^Q \leq \int_{\delta(x,y)<r} d\mu(y) \leq Br^Q \tag{20}$$

Let

$$K_\alpha(x,y) = \frac{1}{\delta(x,y)^{Q-\alpha}} \tag{21}$$

(where $0 < \alpha < Q/2$) and E_{K_α} the associated Banach space (defined in Proposition 1). Let $\mathcal{I}_\alpha$ be the Riesz operator asociated K_α :

$$\mathcal{I}_\alpha f(x) = \int_X K_\alpha(x,y) f(y)\ d\mu(y). \tag{22}$$

We define two further linear spaces associated to K_α :

- *the Sobolev space W^α defined by*

$$g \in W^\alpha \Leftrightarrow \exists h \in L^2\ g = \mathcal{I}_\alpha h \tag{23}$$

- *the multiplier space $\mathcal{V}^\alpha$ defined by*

$$f \in \mathcal{V}^\alpha \Leftrightarrow \|f\|_{\mathcal{V}^\alpha} = \big(\sup_{\|h\|_2 \leq 1} \int_X |f(x)|^2 |\mathcal{I}_\alpha h(x)|^2\ d\mu(x) \big)^{1/2} < +\infty \tag{24}$$

  (so that pointwise multiplication by a function in $\mathcal{V}^\alpha$ maps boundedly W^α to L^2).

Then we have (with equivalence of norms) for $0 < \alpha < Q/2$:

$$E_{K_\alpha} = \mathcal{V}^\alpha. \tag{25}$$

4 Where we export our parabolic equations to the land of elliptic equations.

Theorem 3 thus gives us the answer to our question 2. Recall that we have tranformed the "parabolic" Navier-Stokes equation (2)

$$\vec{u} = e^{t\Delta}\vec{u}_0 - B(\vec{u},\vec{u})$$

into the "elliptic" equation (13

$$W(t,x) = W_0(t,x) + C_0 \int_{\mathbb{R}} \int \frac{1}{(t-s)^2 + |x-y|^4} W^2(s,y)\, dy\, dx.$$

which we interpret as

$$W = W_0 + \mathcal{J}_1(W^2) \tag{26}$$

where $\mathcal{J}_1$ is a generalized Riesz potential on the (parabolic) space of homogeneous type $\mathbb{R} \times \mathbb{R}^3$:

- quasi-norm : $\rho(t,x) = (t^2 + |x|^4)^{1/4}$
- dimension : $\iint_{B((t,x),R)} ds\, dy = cR^5$
- Riesz potential :

$$\mathcal{J}_1 f(t,x) = \iint_{\mathbb{R}\times\mathbb{R}^3} \frac{C_0}{\rho(t-s,x-y)^{5-1}} f(s,y)\, ds\, dy$$

Answer to Question 2 is then the following one [7] :

Theorem 4
Let $W_0 \geq 0$. Then the following assertions are equivalent :

1. *for $\epsilon > 0$ small enough, we have a solution to the equation $u_\epsilon = \epsilon W_0 + \mathcal{J}_1(u_\epsilon^2)$*
2. *W_0 satisfies the inequality :*

$$\iint_{\mathbb{R}\times\mathbb{R}^3} W_0^2(t,x)(\mathcal{J}_1 f(t,x))^2\, dt\, dx \leq C \iint_{\mathbb{R}\times\mathbb{R}^3} f^2(t,x)\, dt\, dx$$

3. *W_0 is a multiplier from the Sobolev space $\dot{H}^{\frac{1}{2},1}_{t,x}$ to L^2 :*

$$\iint_{\mathbb{R}\times\mathbb{R}^3} W_0^2(t,x) f^2(t,x)\, dt\, dx \leq C \iint_{\mathbb{R}\times\mathbb{R}^3} |\Lambda f(t,x)|^2\, dt\, dx$$

where $\Lambda = (-\partial_t^2)^{1/4} + (-\Delta_x)^{1/2}$.

5 Parabolic Morrey spaces and Triebel–Lizorkin-Morrey spaces, and other examples.

Obviously, we have a formal answer to our Question 3 : the good space for initial value should be the space of (divergence-free) vector fields such that $1_{t>0}|e^{t,\Delta}\vec{u}_0|$ is a multiplier from the Sobolev space $\dot{H}^{\frac{1}{2},1}_{t,x}$ to L^2. The problem is that this space is clearly not a classical space of functional analysis, so that, in a way, our answer is tautological : an initial value is good if it is a good initial value, whatever it actually means ...

If we want to get a better insight into what would be a good initial value, we may use a variant of the Fefferman–Phong inequality. We are thus going to compare our space of singular multipliers

$$\mathcal{W} = \mathcal{M}(\dot{H}^{\frac{1}{2},1}_{t,x} \mapsto L^2)$$

to parabolic Morrey spaces $\dot{M}^{p,q}(\mathbb{R}\times\mathbb{R}^3)$ $(1 < p \leq q < +\infty)$:

$$f \in \dot{M}^{p,q}(\mathbb{R}\times\mathbb{R}^3) \Leftrightarrow \sup_{R>0,(t,x)\in\mathbb{R}\times\mathbb{R}^3} R^{5(\frac{p}{q}-1)} \int_{\rho(t-s,x-y)<R} |f(s,y)|^p\, ds\, dy < +\infty$$

Theorem 5
For $2 < p \leq 5$, we have

$$\dot{M}^{p,5}(\mathbb{R}\times\mathbb{R}^3) \subset \mathcal{W} \subset \dot{M}^{2,5}(\mathbb{R}\times\mathbb{R}^3)$$

Then, a better (but partial) answer to our Question 3 would be to find a Banach space Y of measurable functions on $\mathbb{R}\times\mathbb{R}^3$ such that $Y \subset \mathcal{W}$ and to characterize the associated (maximal) Banach space X such that $\|1_{t>0}|e^{t,\Delta}\vec{u}_0|\|_Y \leq C\|\vec{u}_0\|_X$

We may state some classical results in Navier–Stokes theory and check how we may easily show that they obey to our formalism :

- the solutions of Fabes, Jones and Rivière [2] belong to the space $Y = L^p_t L^q_x$ with $\frac{2}{p}+\frac{3}{q}=1$ and $3 \leq q < +\infty$; we have $Y \subset \dot{M}^{\min(p,q),5}(\mathbb{R}\times\mathbb{R}^3) \subset \mathcal{W}$; the associated space is the Besov space $X = \dot{B}^{-\frac{2}{p}}_{q,p}$
- let us consider the limit case $p=+\infty$ and $q=3$: the solutions of Kato [5] belong to the space $Y = L^\infty_t L^3_x$; we have $Y \subset \dot{M}^{3,5}(\mathbb{R}\times\mathbb{R}^3) \subset \mathcal{W}$ and the associated space is $X = L^3$

- we may change the order of integration with respect to time and space. The space $Y = L^q_x L^p_t$ with $\frac{2}{p} + \frac{3}{q} = 1$ and $3 \leq q < +\infty$ will satisfy $Y \subset \dot{M}^{\min(p,q),5}(\mathbb{R} \times \mathbb{R}^3) \subset \mathcal{W}$ and the associated space is the Triebel–Lizorkin space $X = \dot{F}^{-\frac{2}{p}}_{q,p}$

- in the limit case $p = +\infty$ and $q = 3$, we find the solutions of Calderón [1] that belong to the space $Y = L^3_x L^\infty_t$; we have $L^3_x L^\infty_t \subset \dot{M}^{3,5}(\mathbb{R} \times \mathbb{R}^3) \subset \mathcal{W}$ and the associated space is $X = L^3$

Further examples are discussed in [7] and [8]. Of course, one should be interested to understand as much as possible which space X corresponds to the (strange) space $Y = \mathcal{W}$. A close approach should be the investigation of the parabolic Morrey spaces. In this case, one recover some already known spaces :

- We have $\mathcal{W} \subset \dot{M}^{2,5}(\mathbb{R} \times \mathbb{R}^3)$. It is worth noticing that the associated Banach space to $Y = \dot{M}^{2,5}(\mathbb{R} \times \mathbb{R}^3)$ is just the Koch and Tararu space $X = BMO^{-1}$ [6].

- On the other hand, we have $\dot{M}^{p,5}(\mathbb{R} \times \mathbb{R}^3) \subset \mathcal{W}$ when $2 < p \leq 5$. The associated Banach space to $Y = \dot{M}^{p,5}(\mathbb{R} \times \mathbb{R}^3)$ $(2 < p \leq 5)$ belongs to the scale of Triebel–Lizorkin–Morrey spaces studied by Sickel, Yang and Yuan [10] : $X = \dot{F}^{-\frac{2}{p},\frac{1}{p}-\frac{1}{q}}_{p,p}$ with $\frac{2}{p} + \frac{3}{q} = 1$. It might be the first "natural" setting where those spaces appear.

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